The Classical Relativistic Radiation-Reaction Problem for Point Charges

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Outline of the Talk

“Matter is made of atoms” becomes accepted wisdom

By the end of the 19th century the atomistic explanation of the material universe had reached such a widespread acceptance among physicists that **Hilbert** in 1900 at the International Congress of Mathematicians proposed as his **6th problem** to *lay the rigorous mathematical foundations of the macroscopic continuum laws of physics in terms of the Newtonian dynamics of a huge number of atoms*.

(Obviously this is an open-ended project, not a clearly limited problem like Hilbert's problem 8a: The Riemann Hypothesis.)

In the 125 years hence mathematical physicists have made impressive progress on Hilbert's 6th problem with **hard-sphere atoms** by deriving the Maxwell-Boltzmann kinetic equation for dilute gases, and from it the Navier–Stokes equations of fluids!

I next pave the ground for what we are after by formulating a

Modernized Newtonian Version of Hilbert's 6th *problem*.

Modernized Newtonian Version of Hilbert's 6^{th} problem

Assume the matter in our world is made of $N \gg 1$ charged, massive **point particles** that belong to 91 different types, having charges Ze with $Z \in \{-1, 1, 2, \cdots, \not{4}\not{3}, \cdots, \not{6}\not{1}, \cdots, 92\}$, representing electrons and the "stable" nuclei with pertinent typical masses. The particles move through three-dimensional Euclidean space as time goes on, according to Newton's laws of motion with pairwise forces between particles given by:

- electrical Coulomb forces $(\propto \pm 1/r^2)$,
- gravitational Newton forces $(\propto -1/r^2)$,
- stabilizing repulsive forces at atomic range $(\propto e^{-r}/r^3)$.

Q: To which extent does this microscopic model account reasonably accurately for material phenomena in our world?

Obviously this model will fail to capture many phenomena, but the question is not how bad it is. The question is: **How good?**

It sets a reference standard to which one has to aspire to!

Modernized Newtonian Version of Hilbert's 6^{th} problem

A: First of all, mathematically the dynamical model is globally in time well-posed as initial value problem! (As good as it can get!)
This reduces the modernized 6^{th} problem to the following:
Q′ If the initial data for the N particles are chosen to represent the state of matter "today," do the dynamical equations correctly predict the state of matter "tomorrow"? NB: Matter is clustered!
A′: This model captures qualitatively, and to some extent also quantitatively accurately the behavior of physical matter "as we know it" for $2 \leq N \leq 10^{55}$. Such as:

- Electrons and nuclei bind to form atoms;
- Atoms bind to form molecules, etc.
- Celestial-size ground states exist and are $\approx$ spherical;
- Two such ground states carry out $\approx$ Kepler motions;
- Expected: kinetic equations of plasma; elastic solids; etc.

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Quest for a Relativistic Version of Hilbert's 6^{th} Problem: The Einstein–Infeld–Hoffmann Legacy

In 1938, Einstein, Infeld, and Hoffmann published their brave attempt at extracting the equations of motion of N uncharged point particles from Einstein's vacuum field equations for spacetimes with singularities,

$$G^{\mu\nu} = 0 \text{ away from singularities}$$

Subsequently, Wallace, a student of Infeld, extended the EIH work to charged point particles, attempting to obtain equations of motion for the particles from the Einstein–Maxwell system

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \text{ away from singularities,}$$

where $T^{\mu\nu}$ is the energy-momentum-stress tensor for the Maxwell–Lorentz fields away from the point charges.
Unfortunately their papers exhibit VOODOO mathematics!

Quest for a Relativistic Version of Hilbert's 6^{th} Problem

TO TAKE OFF FROM THE GROUND ONE NEEDS:

A well-posed *relativistically covariant* joint initial value problem *for the motion of massive charged point particles and evolution of the electromagnetic and gravitational fields they generate*.

The Classical Radiation-Reaction Problem has stood in the way!

"When people realized there is a problem, quantum physics was invented. Then everyone started doing quantum physics, and the problem was forgotten. Yet, the problem still exists!"

(**Detlef Dürr**, private communication, mid 1980s)

Outlook on what's reported in the remainder of this talk

We report on a well-posed IVP of a special-relativistic classical electrodynamics of N point charges in a BLTP vacuum that we have established in joint work with Shadi **Tahvildar-Zadeh.** Jointly with Holly **Carley**, Ryan **McGuigan**, and Lilit **Sarsyan** we have begun the evaluation of BLTP electrodynamics for a single point charge moving along a constant applied **E** field. We have laid out the stepping stones for taking the Vlasov limit of BLTP electrodynamics to get the Vlasov-MBLTP equations of the kinetic theory of plasma, and its radiation-reaction corrected generalization (joint with Yves **Elskens**). We are confident that the Vlasov limit $N \to \infty$ can be rigorously established for the BLTP electrodynamics, and also the subsequent **singular limit** $\varkappa \to \infty$ to obtain **the relativistic Vlasov-Maxwell system.** Jointly with Annegret **Burtscher** and **Tahvildar-Zadeh** we have begun to generalize our work to the **general-relativistic** setting. This is still in its baby phase.

My Co-Author on Rigorous BLTP Electrodynamics

The special-relativistic joint **I**nitial-**V**alue **P**roblem for the motion of N point charges and the evolution of their electromagnetic Maxwell fields in a BLTP vacuum is a joint work with

A. Shadi Tahvildar-Zadeh

My Co-Authors on BLTP motion in a capacitor **E** field

The straight motion of a point charge along a constant capacitor field to $O(\varkappa^3)$ included is joint work with

Holly **Carley**

and to $O(\varkappa^4)$ included it is joint work with

Ryan **McGuigan**

Holly

Ryan

My Co-Author on Relativistic Kinetic Plasma Theory

The road map for a derivation of special-relativistic kinetic plasma theory from BLTP electrodynamics is joint work with

Yves **Elskens**

My Co-Authors on the EIH legacy

The general-relativistic Initial-Value Problem for a single point charge at rest and its gravitational and electrical fields is joint work with Shadi **Tahvildar-Zadeh** and Annegret **Burtscher**.

Shadi

Annegret

As for the details, ...

The classical radiation-reaction problem was encountered before the advent of special-relativity theory, and therefore before the advent of general-relativity theory. Thus, if we cannot overcome it in a special-relativistic setting, we presumably cannot overcome it in a general-relativistic setting either!
In the following we review the main attempts to establish a mathematically consistent classical special-relativistic electrodynamics with point charges and their electromagnetic interactions, culminating with our own recent contributions.
Subsequently we address the general-relativistic analog, in which also gravitational interactions enter. This has remained a vastly open problem, but slowly progress is being made!

The failed proto-type: Lorentz electrodynamics

The special-relativistic equations of particle motion

- Einstein–Lorentz–Poincaré velocity-momentum relation

$$\dot{\mathbf{q}}_k(t) = \frac{1}{m_k}\frac{\mathbf{p}_k(t)}{\sqrt{1+\frac{|\mathbf{p}_k(t)|^2}{m_k^2c^2}}}; \quad m_k \neq 0$$

- Newton's law for the rate of change of momentum

$$\dot{\mathbf{p}}_k(t) = \mathbf{f}_k(t)$$

- Lorentz' law for the electromagnetic force

$$\mathbf{f}_k^{\mathrm{Lor}}(t) = e_k\left[\mathbf{E}(t,\mathbf{q}_k(t)) + \tfrac{1}{c}\dot{\mathbf{q}}_k(t)\times\mathbf{B}(t,\mathbf{q}_k(t))\right]$$

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The special-relativistic Maxwell–Lorentz field equations:

- The evolution equations:

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$$\partial_t \mathbf{E}(t,\mathbf{s}) = +c\nabla\times\mathbf{B}(t,\mathbf{s}) - 4\pi\sum_k e_k\delta_{\mathbf{q}_k(t)}(\mathbf{s})\dot{\mathbf{q}}_k(t),$$

- The constraint equations:

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- N.B.: Constraint equations restrict only field initial data.

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- Symbolically the equations of Lorentz Electrodynamics seem to pose a joint Cauchy problem for positions $\mathbf{q}_k(t)$ and momenta $\mathbf{p}_k(t)$, and for the fields $\mathbf{B}(t,\mathbf{s})$ and $\mathbf{E}(t,\mathbf{s})$, with initial data constrained by the divergence equations for the fields, and by the speed limit $|\dot{\mathbf{q}}_k(0)| < c$.
- However, $\mathbf{E}(t,\mathbf{q}_k(t))$ & $\mathbf{B}(t,\mathbf{q}_k(t))$ "infinite in all directions"
- $\mathbf{f}_k^{\mathrm{Lor}}(t)$ can be "defined" through averaging (very popular!), but result depends on how the averaging is done.
- Deckert and Hartenstein: Typically, field singularities will propagate along forward light cones of initial positions.
- Also, fields diverge strongly at particle world lines!
  $\longrightarrow$ No meaningful energy-momentum conservation law!
- **UPSHOT: Lorentz Electrodynamics with point charges *is not well-definable*!**

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Schwarzschild–Synge / Fokker–Tetrode ?

Radical proposal: Purge "self" forces & "sacrifice Cauchy"!
In 1903, Karl **Schwarzschild** proposed a law of motion using the RETARDED force of a moving point charge on another, computed with the **Liénard–Wiechert** "elementary fields." After the advent of Relativity Theory, only the kinetic term in Schwarzschild's action principle needed to be changed to account for the relativistic relationship between mechanical momentum and velocity to yield a special-relativistic model, studied in 1940 by John **Synge**.
A more radical variation on this theme was proposed by Hugo M. **Tetrode** (1922) and Adrien D. **Fokker** (1929) who used RETARDED and ADVANCED forces!
$\longrightarrow$ **None of these models poses an IVP**!
$\longrightarrow$ **None of these models obeys "actio = reactio"**!

Wheeler–Feynman ?

In the late 1940s, Richard P. **Feynman** and John A. **Wheeler** picked up on this theme, but imposed a SELECTION RULE: *ABSORBER PRINCIPLE* replaces all contributions from the future by those of the past, plus a third-order time derivative of the positions at the current time.

$\longrightarrow$ **Again NOT an IVP**!

$\longrightarrow$ **Still no "actio = reactio"**!

Rigorous Results for the **S-S**, **F-T**, and **W-F** Models

G. **Bauer**, “Ein Existenzsatz für die WF-Elektrodynamik” (1997).

G. **Bauer**, D.-A. **Deckert**, D. **Dürr**, “On the existence of dynamics in Wheeler–Feynman electromagnetism,” ZAMP, 1–38, (2013).

G. **Bauer**, D.-A. **Deckert**, D. **Dürr**, G. **Hinrichs**, “On irreversibility and radiation in classical electrodynamics of point particles,” J. Statist. Phys. **154**:610–622 (2014).

D.-A. **Deckert** and N. **Vona**, “Delay equations of the Wheeler-Feynman type,” J. Math. Sci. **202**:623–636 (2014).

G. **Bauer**, D.-A. **Deckert**, D. **Dürr**, G. **Hinrichs**: “Global solutions to the electrodynamic two-body problem on a straight line,” ZAMP **68** (2017).

(N.B. Most results are for the FT model)

Gustav Mie's legacy: Generalized EM Vacuum Laws

Meantime, at the IVP front ...

1912/1913: Gustav **Mie** inaugurates nonlinear classical electromagnetic field theory $\Longleftrightarrow$ Introducing non-Maxwellian "laws of the pure ether."

N.B.: **Mie** purges point charges from the theory; seeks "field solitons"
Mie's program was picked up by Max **Born**, David **Hilbert**, Hermann **Weyl**

In 1933 Max **Born** puts the point charges back in. His insight:
Non-Maxwellian vacua can produce integrable energy-momentum densities for the electromagnetic fields of point charges.
In 1934 Leo **Infeld** joins **Born** in his project.

In 1940 Fritz **Bopp** replaces the *nonlinear* **Born-Infeld** model by a *linear* higher order derivative model.
In 1941 the same linear model is proposed by Alfred **Landé**; subsequently collaboration on it with Llewellyn **Thomas.**
Subsequently Boris **Podolsky** picks up on the work of **Landé–Thomas**.

Pre-metric Maxwell–Lorentz field equations (for free!)

- Minkowski spacetime with Lorentz frame $(t, \mathbf{s})$, where $\mathbf{s} \in \mathbb{R}^3$ is a space vector and $t \in \mathbb{R}$ an instant of time.
- N timelike world-lines of moving charged point particles (NB: timelike $\Leftrightarrow |\dot{\mathbf{q}}_k(t)| < c$).
- Poincaré's lemma now implies the existence of fields $\mathbf{D}$, $\mathbf{H}$, $\mathbf{B}$, $\mathbf{E}$ satisfying:
  - The evolution equations

$$\partial_t \mathbf{B}(t, \mathbf{s}) = -c\nabla\times\mathbf{E}(t, \mathbf{s})$$

$$\partial_t \mathbf{D}(t, \mathbf{s}) = +c\nabla\times\mathbf{H}(t, \mathbf{s}) - 4\pi \sum_{k=1}^{N} e_k \delta_{\mathbf{q}_k(t)}(\mathbf{s})\dot{\mathbf{q}}_k(t)$$

  - The constraint equations (restrict only the field initial data)

$$\nabla\cdot\mathbf{B}(t, \mathbf{s}) = 0$$

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$$\nabla \cdot \mathbf{B}(t, \mathbf{s}) = 0$$
$$\nabla \cdot \mathbf{D}(t, \mathbf{s}) = 4\pi \sum_{k=1}^{N} e_k \delta_{\mathbf{q}_k(t)}(\mathbf{s})$$

Electromagnetic Vacuum Law needs to be supplied!

A map $(\mathbf{B},\mathbf{D}) \mapsto (\mathbf{H},\mathbf{E})$ defines the electromagnetic vacuum!

- Maxwell–Lorentz law

$$\mathbf{H} = \mathbf{B} \qquad \& \qquad \mathbf{E} = \mathbf{D}$$

- Born–Infeld law

$$\mathbf{H} = \frac{\mathbf{B} - \frac{1}{b^2}\mathbf{D}\times(\mathbf{D}\times\mathbf{B})}{\sqrt{1 + \frac{1}{b^2}(|\mathbf{B}|^2 + |\mathbf{D}|^2) + \frac{1}{b^4}|\mathbf{B}\times\mathbf{D}|^2}}$$

$$\mathbf{E} = \frac{\mathbf{D} - \frac{1}{b^2}\mathbf{B}\times(\mathbf{B}\times\mathbf{D})}{\sqrt{1 + \frac{1}{b^2}(|\mathbf{B}|^2 + |\mathbf{D}|^2) + \frac{1}{b^4}|\mathbf{B}\times\mathbf{D}|^2}}$$

- Bopp–Landé–Thomas–Podolsky law (N.B.: $\square := \frac{1}{c^2}\partial_t^2 - \Delta$)

$$\mathbf{H}(t,\mathbf{s}) = \left(1 + \varkappa^{-2}\square\right)\mathbf{B}(t,\mathbf{s})$$

$$\mathbf{D}(t,\mathbf{s}) = \left(1 + \varkappa^{-2}\square\right)\mathbf{E}(t,\mathbf{s})\,.$$

- NB: BI $\to$ ML as $b \to \infty$; BLTP $\to$ ML as $\varkappa \to \infty$

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The total Lorentz force is still ill-defined

Neither Born–Infeld nor Bopp, or Landé–Thomas, or Podolsky seem to have realized that the fields $\mathbf{B}(t,\mathbf{s})$ and $\mathbf{E}(t,\mathbf{s})$ generated by a moving point charge are still undefined at the location $\mathbf{q}(t)$ of the point charge.

Even though the Maxwell fields $\mathbf{B}(t,\mathbf{s})$ and $\mathbf{E}(t,\mathbf{s})$ in a BI or a BLTP vacuum are believed not to blow up in magnitude (for MBLTP fields that has been proven), the limit of these fields as $\mathbf{s} \to \mathbf{q}(t)$ does not exist, and so the field singularities cannot be removed. Again, averaging over the neighborhood can produce any vector that interpolates between the extreme possibilities.

The Lorentz "self"-force remains not well-definable!

Poincaré's definition of the EM force

Poincaré's electromagnetic force when only a single charge is present:

$$\mathbf{f}(t) := -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \mathbf{\Pi}(t, \mathbf{s})d^3s,$$

where $\mathbf{\Pi}$ is the momentum density of the electromagnetic field.

NB: Poincaré died before the Born–Infeld and Bopp proposals appeared. He used the Maxwell–Lorentz field equations and replaced point charges by smeared-out charges (similar to Abraham or Lorentz) and later postulated Poincaré stresses to keep the particle structure stable.

Poincaré's definition of the electromagnetic force can be utilized for **point charges** in the Bopp–Landé–Thomas–Podolsky electromagnetic vacuum, and *presumably* also in the Born–Infeld vacuum.

Electromagnetic Field Momenta

ML, MBI, and MBLTP field momentum densities $\mathbf{\Pi}$:

- For ML and for MBI field equations

$$4\pi\mathbf{\Pi} = \mathbf{D}\times\mathbf{B}$$

- For MBLTP field equations

$$4\pi\mathbf{\Pi} = \mathbf{D}\times\mathbf{B} + \mathbf{E}\times\mathbf{H} - \mathbf{E}\times\mathbf{B} - \varkappa^{-2}(\nabla\cdot\mathbf{E})(\nabla\times\mathbf{B} - \varkappa\dot{\mathbf{E}})$$

- $\mathbf{\Pi}(t,\mathbf{s})$ is $L^1_{loc}(\mathbb{R}^3)$ about each $\mathbf{q}(t)$ for MBLTP fields (KTZ). (Expected for MBI fields, but surely FALSE for ML fields!)

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BLTP Electrodynamics with a single point charge

Momentum Conservation $\longrightarrow$ Equation of Motion (1 pt charge)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \mathbf{\Pi}(t,\mathbf{s})d^3s \qquad (*)$$

- With BLTP law: The fields $\mathbf{B}, \mathbf{D}, \mathbf{E}, \dot{\mathbf{E}}$ (and $\mathbf{H}$) at $(t, \mathbf{s})$ depend on $\mathbf{q}(\cdot)$, $\mathbf{p}(\cdot)$, while $\mathbf{D}$ & $\mathbf{H}$ depend also on $\mathbf{a}(\cdot)$ (**linearly**), and then $(*)$ is equivalent to a linear **Volterra** integral equation of the second kind for $\mathbf{a} = \mathbf{a}[\mathbf{q}, \mathbf{p}]$
- Integration over time leads to the fixed point equations

$$\mathbf{q}(t) = \mathbf{q}(0) + \frac{1}{m}\int_0^t \frac{\mathbf{p}}{\sqrt{1+\frac{|\mathbf{p}|^2}{m^2c^2}}}(\tilde{t})d\tilde{t} \qquad =: Q_t(\mathbf{q}(\cdot), \mathbf{p}(\cdot))$$

$$\mathbf{p}(t) = \mathbf{p}(0) - \int_{\mathbb{R}^3} \left(\mathbf{\Pi}(t,\mathbf{s}) - \mathbf{\Pi}(0,\mathbf{s})\right) d^3s =: P_t(\mathbf{q}(\cdot), \mathbf{p}(\cdot))$$

- Well-posedness if $(Q_\bullet, P_\bullet)(\,\cdot\,,\,\cdot\,)$ is a Lipschitz Map.

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The key proposition (for a single point charge)

Proposition (KTZ) *Given $C^{0,1}$ maps $t \mapsto \mathbf{q}(t)$ and $t \mapsto \mathbf{p}(t)$, with Lip$(\mathbf{q}) = v$, Lip$(\mathbf{v}) = a$, and $|\mathbf{v}(t)| \leq v < c$, the Volterra equation as a fixed point map has a unique C^0 solution $t \mapsto \mathbf{a}(t) = \alpha[\mathbf{q}(\,\cdot\,), \mathbf{p}(\,\cdot\,)](t)$. Moreover, the solution depends Lipschitz continuously on the maps $t \mapsto \mathbf{q}(t)$ and $t \mapsto \mathbf{p}(t)$.*

The proof takes several dozen pages of careful estimates, but at the end of the day it all pans out! The well-posedness result for the joint initial value problem of MBLTP fields and their point charge sources is a corollary of the above Proposition.

The Volterra equation for the BLTP acceleration

$$\mathbf{a} = W[\mathbf{p}] \cdot \Big(\mathbf{f}^{\text{vac}}[\mathbf{q},\mathbf{v}] + \mathbf{f}^{\text{source}}[\mathbf{q},\mathbf{p};\mathbf{a}]\Big)$$

where

$$W[\mathbf{p}] := \operatorname{sign}(m)\frac{1}{\sqrt{m^2c^2+|\mathbf{p}|^2}}\left[\mathbf{I} - \frac{\mathbf{p}\otimes\mathbf{p}}{m^2c^2+|\mathbf{p}|^2}\right]$$

and

$$\mathbf{f}^{\text{vac}}[\mathbf{q},\mathbf{v}](t) \equiv e\left[\mathbf{E}^{\text{vac}}(t,\mathbf{q}(t)) + \tfrac{1}{c}\mathbf{v}(t)\times\mathbf{B}^{\text{vac}}(t,\mathbf{q}(t))\right]$$

with

$$\mathbf{v} = \frac{1}{m}\frac{\mathbf{p}}{\sqrt{1+\frac{|\mathbf{p}|^2}{m^2c^2}}}; \qquad m \neq 0.$$

and

$$\mathbf{f}^{\text{source}}[\mathbf{q},\mathbf{p};\mathbf{a}](t) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\mathbf{\Pi}^{\text{source}}(t,\mathbf{s})d^3s$$

The Volterra equation for the BLTP acceleration

$\mathbf{f}^{\text{source}}[\mathbf{q},\mathbf{v};\mathbf{a}]$ is the "self" force in BLTP electrodynamics

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$$= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{ct}(\mathbf{q}_0)}\left(\mathbf{\Pi}(t,\mathbf{s}) - \mathbf{\Pi}(0,\mathbf{s}-\mathbf{q}_0-\mathbf{v}_0 t)\right)d^3s$$

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$$\begin{aligned}
\mathbf{f}[\mathbf{q},\mathbf{v};\mathbf{a}](t) &= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\mathbf{\Pi}(t,\mathbf{s})d^3s \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{ct}(\mathbf{q}_0)}\left(\mathbf{\Pi}(t,\mathbf{s})-\mathbf{\Pi}(0,\mathbf{s}-\mathbf{q}_0-\mathbf{v}_0 t)\right)d^3s \\
&= \frac{e^2}{4\pi}\Bigg[-\mathbf{Z}^{[2]}_{\boldsymbol{\xi}}(t,t)+\mathbf{Z}^{[2]}_{\boldsymbol{\xi}^\circ}(t,t) \\
&\quad -\sum_{0\le k\le 1}(2-k)\int_0^t\left[\mathbf{Z}^{[k]}_{\boldsymbol{\xi}}(t,t^{\mathrm{r}})-\mathbf{Z}^{[k]}_{\boldsymbol{\xi}^\circ}(t,t^{\mathrm{r}})\right](t-t^{\mathrm{r}})^{1-k}\mathrm{d}t^{\mathrm{r}} \\
&\quad -\sum_{0\le k\le 2}\int_0^t\left[\frac{\partial}{\partial t}\mathbf{Z}^{[k]}_{\boldsymbol{\xi}}(t,t^{\mathrm{r}})-\frac{\partial}{\partial t}\mathbf{Z}^{[k]}_{\boldsymbol{\xi}^\circ}(t,t^{\mathrm{r}})\right](t-t^{\mathrm{r}})^{2-k}\mathrm{d}t^{\mathrm{r}}\Bigg]
\end{aligned}$$

where $\xi(t)\equiv(\mathbf{q},\mathbf{v},\mathbf{a})(t)$, and $\xi^\circ(t)\equiv(\mathbf{q}_0+\mathbf{v}_0t,\mathbf{v}_0,\mathbf{0})$, and ...

The Volterra equation for the BLTP acceleration

$$\mathbf{Z}_{\boldsymbol{\xi}}^{[k]}(t, t^{\mathrm{r}}) =$$
$$\int_0^{2\pi}\int_0^{\pi} (1 - |\boldsymbol{v}(t^{\mathrm{r}})| \cos\vartheta)\, \boldsymbol{\pi}_{\boldsymbol{\xi}}^{[k]}\big(t, \mathbf{q}(t^{\mathrm{r}}) + c(t - t^{\mathrm{r}})\boldsymbol{n}\big) \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi\,,$$

with

$$\boldsymbol{n} = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$$

and where, with $\big|_{\mathrm{ret}}$ meaning that $\mathbf{q}(\tilde{t})$, $\boldsymbol{v}(\tilde{t})$, $\boldsymbol{a}(\tilde{t})$ are evaluated at $\tilde{t} = t_{\boldsymbol{\xi}}^{\mathrm{ret}}(t, \mathbf{s})$, we have ...

The Volterra equation for the BLTP acceleration

$$
\begin{aligned}
\pi_{\xi}^{[0]}(t,\mathbf{s}) = &- \varkappa^4 \frac{1}{4}\left[\frac{\big(\boldsymbol{n}(\mathbf{q},\mathbf{s}) - \boldsymbol{v}\big)\times\big(\boldsymbol{v}\times\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)}{\big(1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^2}\right]_{\mathrm{ret}} \\
&+ \varkappa^4 \frac{1}{2}\left[\frac{\boldsymbol{n}(\mathbf{q},\mathbf{s}) - \boldsymbol{v}}{1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})}\right]_{\mathrm{ret}} \times \int_{-\infty}^{t_{\xi}^{\mathrm{ret}}(t,\mathbf{s})} \boldsymbol{v}(t')\times\mathbf{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t' \\
&- \varkappa^4 \frac{1}{2}\left[\frac{\boldsymbol{v}\times\boldsymbol{n}(\mathbf{q},\mathbf{s})}{1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})}\right]_{\mathrm{ret}} \times \int_{-\infty}^{t_{\xi}^{\mathrm{ret}}(t,\mathbf{s})} \mathbf{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t' \\
&- \varkappa^4 \int_{-\infty}^{t_{\xi}^{\mathrm{ret}}(t,\mathbf{s})} \mathbf{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t' \times \int_{-\infty}^{t_{\xi}^{\mathrm{ret}}(t,\mathbf{s})} \boldsymbol{v}(t')\times\mathbf{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t' \\
&- \varkappa^4 \int_{-\infty}^{t_{\xi}^{\mathrm{ret}}(t,\mathbf{s})} \mathrm{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t' \int_{-\infty}^{t_{\xi}^{\mathrm{ret}}(t,\mathbf{s})} \mathrm{K}_{\xi}(t',t,\mathbf{s})\boldsymbol{v}(t')\mathrm{d}t'
\end{aligned}
$$

The Volterra equation for the BLTP acceleration

$$\begin{aligned}\pi_{\xi}^{[1]}(t,\mathbf{s}) = &-\varkappa^2\left[\boldsymbol{n}(\mathbf{q},\mathbf{s})\frac{(\boldsymbol{n}(\mathbf{q},\mathbf{s})\times[(\boldsymbol{n}(\mathbf{q},\mathbf{s})-\boldsymbol{v})\times\boldsymbol{a}])\cdot\boldsymbol{v}}{\big(1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^4}+\boldsymbol{n}(\mathbf{q},\mathbf{s})\times\frac{\big(\boldsymbol{n}(\mathbf{q},\mathbf{s})-\boldsymbol{v}\big)\times\boldsymbol{a}}{2\big(1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^3}\right]_{\mathrm{ret}}\\
&-\varkappa^2\left[\boldsymbol{n}(\mathbf{q},\mathbf{s})\times\frac{\big(\boldsymbol{n}(\mathbf{q},\mathbf{s})-\boldsymbol{v}\big)\times\boldsymbol{a}}{\big(1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^3}\right]_{\mathrm{ret}}\times\int_{-\infty}^{t_{\xi}^{\mathrm{ret}}(t,\mathbf{s})}\boldsymbol{v}(t')\times\mathbf{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t'\\
&+\varkappa^2\left[\boldsymbol{n}(\mathbf{q},\mathbf{s})\times\left[\boldsymbol{n}(\mathbf{q},\mathbf{s})\times\frac{\big(\boldsymbol{n}(\mathbf{q},\mathbf{s})-\boldsymbol{v}\big)\times\boldsymbol{a}}{\big(1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^3}\right]\right]_{\mathrm{ret}}\times\int_{-\infty}^{t_{\xi}^{\mathrm{ret}}(t,\mathbf{s})}\mathbf{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t'\\
&+\varkappa^3\left[\frac{1}{1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})}\right]_{\mathrm{ret}}\int_{-\infty}^{t_{\xi}^{\mathrm{ret}}(t,\mathbf{s})}\mathrm{K}_{\xi}(t',t,\mathbf{s})\left[\boldsymbol{v}(t_{\xi}^{\mathrm{ret}}(t,\mathbf{s}))+\boldsymbol{v}(t')\right]\mathrm{d}t'\end{aligned}$$

The Volterra equation for the BLTP acceleration

$$\begin{aligned}\pi_\xi^{[2]}(t,s) = &-\varkappa^2\left[\frac{1}{\left(1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\right)^2}\boldsymbol{v}-\left[1-\left|\boldsymbol{v}\right|^2\right]\frac{\left(\boldsymbol{n}(\mathbf{q},\mathbf{s})-\boldsymbol{v}\right)\times\left(\boldsymbol{v}\times\boldsymbol{n}(\mathbf{q},\mathbf{s})\right)}{\left(1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\right)^4}\right]_{\mathrm{ret}}\\
&+\varkappa^2\left[\left[1-\left|\boldsymbol{v}\right|^2\right]\boldsymbol{n}(\mathbf{q},\mathbf{s})\times\frac{\boldsymbol{n}(\mathbf{q},\mathbf{s})-\boldsymbol{v}}{\left(1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\right)^3}\right]_{\mathrm{ret}}\times\int_{-\infty}^{t_\xi^{\mathrm{ret}}(t,\mathbf{s})}\mathbf{K}_\xi(t',t,\mathbf{s})\mathrm{d}t'\\
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with the abbreviations

$$\mathrm{K}_\xi(t',t,\mathbf{s}) := \frac{J_1\left(\varkappa\sqrt{(t-t')^2-|\mathbf{s}-\mathbf{q}(t')|^2}\right)}{\sqrt{(t-t')^2-|\mathbf{s}-\mathbf{q}(t')|^2}},$$

$$\mathbf{K}_\xi(t',t,\mathbf{s}) := \frac{J_2\left(\varkappa\sqrt{(t-t')^2-|\mathbf{s}-\mathbf{q}(t')|^2}\right)}{(t-t')^2-|\mathbf{s}-\mathbf{q}(t')|^2}\left(\mathbf{s}-\mathbf{q}(t')-\boldsymbol{v}(t')(t-t')\right).$$

Thm: BLTP Electrodynamics is well-posed $\forall N$ (KTZ)

We have generalized Poincaré's force definition to N-particles, using hyperbolicity of the field equations. We have proved that:

- The Cauchy problem for MBLTP fields + N point charges:
  - is locally well-posed for admissible initial data & $m \neq 0$;
  - is globally well-posed **if** in a finite time:
    (a) no particle reaches the speed of light,
    (b) no particle reaches infinite acceleration,
    (c) no two particles reach the same location;
  - obeys Energy-Momentum conservation rigorously;
  - rigorously furnishes a "Self"-force.
- MBLTP field theory features the following **oddities**:
  (a) longitudinal electrical waves;
  (b) subluminal transversal electromagnetic wave modes;
  (c) energy functional unbounded below;
  (d) The MBLTP fields $\mathbf{B}, \mathbf{D}, \mathbf{E}, \dot{\mathbf{E}}$ require initial data.
  N.B.: $(\mathbf{B}, \mathbf{D})_0 \mapsto (\mathbf{E}, \dot{\mathbf{E}})_0$ feasible! (max. field energy)

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  - obeys Energy-Momentum conservation rigorously;
  - rigorously furnishes a "Self"-force.
- MBLTP field theory features the following **oddities**:
  (a) longitudinal electrical waves;
  (b) subluminal transversal electromagnetic wave modes;
  (c) energy functional unbounded below;
  (d) The MBLTP fields $\mathbf{B}, \mathbf{D}, \mathbf{E}, \dot{\mathbf{E}}$ require initial data.
  N.B.: $(\mathbf{B}, \mathbf{D})_0 \mapsto (\mathbf{E}, \dot{\mathbf{E}})_0$ feasible! (max. field energy)

Thm: BLTP Electrodynamics is well-posed $\forall N$ (KTZ)

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Publications on rigorous BLTP electrodynamics

The proof of local well-posedness for the joint initial value problem will appear in:
M.K.-H.K. and A.S. Tahvildar-Zadeh, "BLTP Electrodynamics as Initial Value Problem," > 100*pgs*. (still in preparation, 2026)

A summary did appear in:
M.K.-H.K., "Force on a point charge source of the classical electromagnetic field," Phys. Rev. D **100**, 065012 (2019); "Erratum," ibid. **101**, 109901(E) (2020).

The global well-posedness of the scattering problem of a single point charge in BLTP electrodynamics (for a fixed external, compactly supported potential) is shown in:
Vu Hoang, Maria Radosz, Angel Harb, Aaron DeLeon, and Alan Baza, "Radiation reaction in higher-order electrodynamics," J. Math. Phys. **62**, 072901 (2021).

BLTP motion along a constant electric capacitor field

In this setting the Abraham–Lorentz–Dirac; Landau–Lifshitz; Eliezer models yield vanishing radiation-reaction.

In BLTP electrodynamics, radiation-reaction does not vanish.

Expansion in powers of $\varkappa$ up to 3rd order included needed to see a non-vanishing radiation-reaction term:
H.K. **Carley** and M.K.-H.K., in "Physics and the Nature of Reality: Essays in Memory of Detlef Dürr," A. Bassi et al. (Eds.), Springer (2024).

Motion with radiation-reaction force expaned in powers of $\varkappa$ up to 4th order included carried out jointly with Ryan **McGuigan**; publication to appear in Int. J. Mod. Phys. A (2026)

Not yet published: Evaluation of $O(\varkappa^5)$ with Lilit **Sargsyan.**
But one really needs to study the large-$\varkappa$ regime, and that is non-perturbative. (On the to-do list!)

BLTP motion along a constant electric capacitor field

The $O(\varkappa^2)$ contribution to the self-force has been computed to be

$$\mathcal{F}_0^{(2)}(t) = \mathbf{0}.$$

In this problem of straight line motion in a constant external electric field, with the particle starting from rest, the BLTP radiation-reaction force *vanishes identically* at $O(\varkappa^2)$.
The $O(\varkappa^3)$ contribution to the radiation-reaction force for small $\varkappa$ has been computed in to be

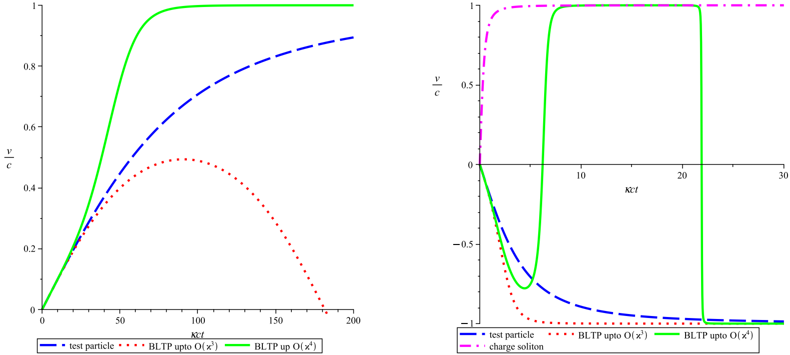
$$\mathcal{F}_0^{(3)}(t) = -\frac{1}{3}\varkappa^3 e^2 \mathbf{q}(t).$$

The $O(\varkappa^4)$ contribution to the radiation-reaction force for small $\varkappa$ reads

$$\mathcal{F}_0^{(4)}(t) = \frac{1}{4}\varkappa^4 e^2 \int_0^t \mathbf{q}(t^r) c \mathrm{d}t^r.$$

BLTP motion along a constant electric capacitor field

NUMERICAL RESULTS ***v*** vs. t FOR THE SMALL $\varkappa$ REGIME.



From McGuigan and Kiessling, to appear 2026.

Special-Relativistic Version of Hilbert's 6^{th} Problem

The Vlasov-Maxwell(-Maxwell) equations are symbolically a singular limit $\varkappa \to \infty$ of the Vlasov-Maxwell-BLTP equations

$$\forall\sigma: \quad \partial_t f_\sigma + \boldsymbol{v}_\sigma\cdot\nabla_{\mathbf{s}} f_\sigma + e_\sigma\left(\mathbf{E} + \tfrac{1}{c}\boldsymbol{v}_\sigma\times\mathbf{B}\right)\cdot\nabla_{\mathbf{p}} f_\sigma = 0\,,$$

$$\boldsymbol{v}_\sigma(\mathbf{p}) = c\mathbf{p}/\sqrt{m_\sigma^2c^2 + |\mathbf{p}|^2}.$$

$$\partial_t\,\mathbf{B}(t,\mathbf{s}) + c\nabla_{\mathbf{s}}\times\mathbf{E}(t,\mathbf{s}) = \mathbf{0}\,,$$

$$\nabla_{\mathbf{s}}\cdot\mathbf{B}(t,\mathbf{s}) = 0\,,$$

$$-\partial_t\,\mathbf{D}(t,\mathbf{s}) + c\nabla_{\mathbf{s}}\times\mathbf{H}(t,\mathbf{s}) = 4\pi\sum_\sigma N_\sigma e_\sigma\int_{\mathbb{R}^3} f_\sigma(t,\mathbf{s},\mathbf{p})\boldsymbol{v}_\sigma(\mathbf{p})\mathrm{d}^3p\,,$$

$$\nabla_{\mathbf{s}}\cdot\mathbf{D}(t,\mathbf{s}) = 4\pi\sum_\sigma N_\sigma e_\sigma\int_{\mathbb{R}^3} f_\sigma(t,\mathbf{s},\mathbf{p})\mathrm{d}^3p,$$

$$\mathbf{D}(t,\mathbf{s}) = (1+\varkappa^{-2}\Box)\mathbf{E}(t,\mathbf{s}) \quad \& \quad \mathbf{H}(t,\mathbf{s}) = (1+\varkappa^{-2}\Box)\mathbf{B}(t,\mathbf{s}).$$

Special-Relativistic Version of Hilbert's 6^{th} Problem

In "Microscopic foundations of kinetic plasma theory: The relativistic Vlasov–Maxwell equations and their radiation-reaction-corrected generalization," J. Statist. Phys. **180**, 749–772 (2020), Y. Elskens and M.K.-H.K. show that in principle the Vlasov-Maxwell-BLTP system should be obtainable as a limit $N_\sigma \to \infty$ of BLTP electrodynamics on suitably short time scales. Through a **subsequent** singular limit $\varkappa \to \infty$ one should be able to obtain the relativistic Vlasov–Maxwell equations that are generally considered to capture relativistic astrophysical plasma phenomena.

Strategy is not via BBGKY hierarchy, but using empirical 1 and 2 point measures. **N.B.: Our strategy is inspired by non-relativistic works of Neunzert (1974) and Dobrushin (1979) who work with regularized Newtonian pair interactions and need only empirical 1 point measures.**

Special-Relativistic Version of Hilbert's 6^{th} Problem

The normalized empirical one-point measure on $\mathbb{R}^6$ of species σ reads,

$$\underline{\triangle}^{(1)}_{N_\sigma}(t,\mathbf{s},\mathbf{p}) := \frac{1}{N_\sigma}\sum_{k=1}^{N_\sigma} \delta_{\mathbf{q}^\sigma_k(t)}(\mathbf{s})\delta_{\mathbf{p}^\sigma_k(t)}(\mathbf{p})\,,$$

and the normalized empirical two-point measure on $\mathbb{R}^6 \times \mathbb{R}^6$ reads

$$\underline{\triangle}^{(2)}_{N_\sigma}(t,\mathbf{s},\mathbf{p},\tilde{t},\tilde{\mathbf{s}},\tilde{\mathbf{p}}) = \frac{1}{N_\sigma(N_\sigma-1)}\sum\sum_{1\leq j\neq k\leq N_\sigma} \delta_{\mathbf{q}^\sigma_j(t)}(\mathbf{s})\delta_{\mathbf{p}^\sigma_j(t)}(\mathbf{p})\delta_{\mathbf{q}^\sigma_k(\tilde{t})}(\tilde{\mathbf{s}})\delta_{\mathbf{p}^\sigma_k(\tilde{t})}(\tilde{\mathbf{p}}).$$

For each σ these two empirical measures jointly satisfy a distributional identity, shown next.

Special-Relativistic Version of Hilbert's 6^{th} Problem

$$
\begin{aligned}
&\partial_t \underline{\triangle}^{(1)}_{N_\sigma}(t,\mathbf{s},\mathbf{p}) + \boldsymbol{v}_\sigma(\mathbf{p})\cdot\partial_{\mathbf{s}}\underline{\triangle}^{(1)}_{N_\sigma}(t,\mathbf{s},\mathbf{p})\\
&\qquad + e_\sigma\left(\mathbf{E}_0(t,\mathbf{s}) + \tfrac{1}{c}\boldsymbol{v}_\sigma(\mathbf{p})\times\mathbf{B}_0(t,\mathbf{s})\right)\cdot\nabla_{\mathbf{p}}\underline{\triangle}^{(1)}_{N_\sigma}(t,\mathbf{s},\mathbf{p})\\
&\qquad + e_\sigma\sum_{\tau\neq\sigma}\left[\mathbf{E}^{(N_\beta)}(t,\mathbf{s}) + \tfrac{1}{c}\boldsymbol{v}_\sigma(\mathbf{p})\times\mathbf{B}^{(N_\beta)}(t,\mathbf{s})\right]\cdot\nabla_{\mathbf{p}}\underline{\triangle}^{(1)}_{N_\sigma}(t,\mathbf{s},\mathbf{p})\\
&+ e_\sigma^2\varkappa^2\tfrac{1}{8\pi}(N_\sigma-1)\iint\left[\frac{\boldsymbol{n}(\tilde{\mathbf{s}},\mathbf{s})-\tilde{\boldsymbol{v}}_\sigma/c}{1-\boldsymbol{n}(\tilde{\mathbf{s}},\mathbf{s})\cdot\tilde{\boldsymbol{v}}_\sigma/c} + \tfrac{1}{c}\boldsymbol{v}_\sigma\times\frac{\tilde{\boldsymbol{v}}_\sigma\times\boldsymbol{n}(\tilde{\mathbf{s}},\mathbf{s})/c}{1-\boldsymbol{n}(\tilde{\mathbf{s}},\mathbf{s})\cdot\tilde{\boldsymbol{v}}_\sigma/c}\right]\cdot\nabla_{\mathbf{p}}\\
&\qquad\qquad\underline{\triangle}^{(2)}_{N_\sigma}(t,\mathbf{s},\mathbf{p},t^{\mathrm{ret}}(t,\mathbf{s}),\tilde{\mathbf{s}},\tilde{\mathbf{p}})\mathrm{d}^3\tilde{p}\,\mathrm{d}^3\tilde{s}\\
&- e_\sigma^2\varkappa^2\tfrac{1}{4\pi}(N_\sigma-1)\int_{-\infty}^{t^{\mathrm{ret}}(t,\mathbf{s})}\iint\left[c\mathbf{K}_{\tilde{\mathbf{s}},\tilde{\boldsymbol{v}}_\sigma}(\tilde{t},t,\mathbf{s}) + \boldsymbol{v}_\sigma\times(\tilde{\boldsymbol{v}}_\sigma\times\mathbf{K}_{\tilde{\mathbf{s}},\tilde{\boldsymbol{v}}_\sigma}(\tilde{t},t,\mathbf{s}))\right]\cdot\nabla_{\mathbf{p}}\\
&\qquad\qquad\underline{\triangle}^{(2)}_{N_\sigma}(t,\mathbf{s},\mathbf{p},\tilde{t},\tilde{\mathbf{s}},\tilde{\mathbf{p}})\mathrm{d}^3\tilde{p}\,\mathrm{d}^3\tilde{s}\,\mathrm{d}\tilde{t}\\
&= \frac{1}{N_\sigma}\sum_{n=1}^{N_\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{B_{ct}(\mathbf{q}^\sigma_n(0))}\left(\mathbf{\Pi}^{\text{field}}_{\sigma,n}(t,\tilde{\mathbf{s}}) - \mathbf{\Pi}^{\text{field}}_{\sigma,n}(0,\tilde{\mathbf{s}}-\overline{\mathbf{q}}^\sigma_n(t))\right)d^3\tilde{s}\right]\cdot\nabla_{\mathbf{p}}\delta_{\mathbf{p}^\sigma_n(t)}(\mathbf{p})\delta_{\mathbf{q}^\sigma_n(t)}(\mathbf{s})\,.
\end{aligned}
$$

Special-Relativistic Version of Hilbert's 6^{th} Problem

Here, $\boldsymbol{n}(\tilde{\mathbf{s}},\mathbf{s}) := \frac{\mathbf{s}-\tilde{\mathbf{s}}}{|\mathbf{s}-\tilde{\mathbf{s}}|}$ for $\tilde{\mathbf{s}} \neq \mathbf{s}$, and where "$|_{\mathrm{ret}}$" means $(\mathbf{q},\boldsymbol{v},\boldsymbol{a}) = (\mathbf{q},\boldsymbol{v},\boldsymbol{a})(t^{\mathrm{ret}})$ with $t^{\mathrm{ret}}(t,\mathbf{s}) < t$ defined implicitly by $c(t-t^{\mathrm{ret}}) = |\mathbf{s}-\mathbf{q}(t^{\mathrm{ret}})|$, with $\mathbf{q}(t)$ continuously extended by straight line motion to $t \leq 0$. The integral kernels are

$$\mathbf{K}_{\tilde{\mathbf{s}},\tilde{\boldsymbol{v}}}(\tilde{t},t,\mathbf{s}) := \frac{J_2\Big(\varkappa\sqrt{c^2(t-\tilde{t})^2-|\mathbf{s}-\tilde{\mathbf{s}}|^2}\Big)}{c^2(t-\tilde{t})^2-|\mathbf{s}-\tilde{\mathbf{s}}|^2}\Big(\mathbf{s}-\tilde{\mathbf{s}}-(t-\tilde{t})\tilde{\boldsymbol{v}}\Big)\,, \qquad (2)$$

Special-Relativistic Version of Hilbert's 6^{th} Problem

For each τ the MBLTP fields $\mathbf{E}^{(N_\beta)}(t,\mathbf{s})$ and $\mathbf{B}^{(N_\beta)}(t,\mathbf{s})$ read

$$\begin{aligned}\mathbf{E}^{(N_\beta)}(t,\mathbf{s}) =& N_\tau e_\tau \varkappa^2 \tfrac{1}{8\pi}\iint \tfrac{\boldsymbol{n}(\tilde{\mathbf{s}},\mathbf{s})-\tilde{\boldsymbol{v}}_\tau/c}{1-\boldsymbol{n}(\tilde{\mathbf{s}},\mathbf{s})\cdot\tilde{\boldsymbol{v}}_\tau/c}\,\triangle^{(1)}_{N_\tau}(t^{\mathrm{ret}}(t,\mathbf{s}),\tilde{\mathbf{s}},\tilde{\mathbf{p}})\mathrm{d}^3\tilde{p}\mathrm{d}^3\tilde{s}\\ &- N_\tau e_\tau \varkappa^2 \tfrac{1}{4\pi}\int_{-\infty}^{t^{\mathrm{ret}}(t,\mathbf{s})}\iint c\mathbf{K}_{\tilde{\mathbf{s}},\tilde{\boldsymbol{v}}_\tau}(\tilde{t},t,\mathbf{s})\triangle^{(1)}_{N_\tau}(\tilde{t},\tilde{\mathbf{s}},\tilde{\mathbf{p}})\mathrm{d}^3\tilde{p}\mathrm{d}^3\tilde{s}\mathrm{d}\tilde{t},\end{aligned}\tag{3}$$

$$\begin{aligned}\mathbf{B}^{(N_\beta)}(t,\mathbf{s}) =& N_\tau e_\tau \varkappa^2 \tfrac{1}{8\pi}\iint \tfrac{\tilde{\boldsymbol{v}}_\tau\times\boldsymbol{n}(\tilde{\mathbf{s}},\mathbf{s})/c}{1-\boldsymbol{n}(\tilde{\mathbf{s}},\mathbf{s})\cdot\tilde{\boldsymbol{v}}_\tau/c}\triangle^{(1)}_{N_\tau}(t^{\mathrm{ret}}(t,\mathbf{s}),\tilde{\mathbf{s}},\tilde{\mathbf{p}})\mathrm{d}^3\tilde{p}\mathrm{d}^3\tilde{s}\\ &- N_\tau e_\tau \varkappa^2 \tfrac{1}{4\pi}\int_{-\infty}^{t^{\mathrm{ret}}(t,\mathbf{s})}\iint \tilde{\boldsymbol{v}}_\tau\times\mathbf{K}_{\tilde{\mathbf{s}},\tilde{\boldsymbol{v}}_\tau}(\tilde{t},t,\mathbf{s})\triangle^{(1)}_{N_\tau}(\tilde{t},\tilde{\mathbf{s}},\tilde{\mathbf{p}})\mathrm{d}^3\tilde{p}\mathrm{d}^3\tilde{s}\mathrm{d}\tilde{t},\end{aligned}\tag{4}$$

while $\mathbf{E}_0(t,\mathbf{s})$ and $\mathbf{B}_0(t,\mathbf{s})$ are vacuum MBLTP fields.

Special-Relativistic Version of Hilbert's 6^{th} Problem

Lastly, the term at r.h.s. is the radiation reaction averaged over all particles of species σ; we have spelled out this term in (ElsKie2020) and here simply abbreviate it as $\mathcal{R}\left[\underline{\triangle}_{N_\sigma}^{(1)}\right](t, \mathbf{s}, \mathbf{p})$. In a nutshell, kinetic theory operates under the assumption that the fine details of the one- and two-particle distributions don't matter for the answers to the questions one has, in the sense that for the practical purpose at hand the microscopically accurate empirical one-point density $\underline{\triangle}_{N_\sigma}^{(1)}$ can be replaced by a smoother, in particular continuous density function $f_\sigma(t, \mathbf{s}, \mathbf{p})$.

Special-Relativistic Version of Hilbert's 6^{th} Problem

Mathematically more precisely, one assumes that $\mathrm{dist}_{KR}(\underline{\triangle}^{(1)}_{N_\sigma} - f_\sigma) < \epsilon$ in a suitable Kantorovich–Rubinstein distance, with ϵ as small as dictated by the error one is willing to tolerate. For sufficiently regular interactions, the empirical two-point density $\underline{\triangle}^{(2)}_{N_\sigma}(t,\mathbf{s},\mathbf{p},\tilde{t},\tilde{\mathbf{s}},\tilde{\mathbf{p}})$ can then be replaced with a continuous product density function $f_\sigma(t,\mathbf{s},\mathbf{p})f_\sigma(\tilde{t},\tilde{\mathbf{s}},\tilde{\mathbf{p}})$, with compatible small error.

In BLTP electrodynamics the interaction is represented by the Green function kernels $\mathbf{K}_{\tilde{\mathbf{s}},\tilde{\boldsymbol{v}}_\tau}(\tilde{t},t,\mathbf{s})$, which are sufficiently regular. The "kinetic approximation" thus results in the following system of transport equations for the $f_\sigma(t,\mathbf{s},\mathbf{p})$:

$$\Big[\partial_t + \boldsymbol{v}_\sigma(\mathbf{p})\cdot\partial_{\mathbf{s}} + e_\sigma\left(\mathbf{E}(t,\mathbf{s}) + \tfrac{1}{c}\boldsymbol{v}_\sigma(\mathbf{p})\times\mathbf{B}(t,\mathbf{s})\right)\cdot\nabla_{\mathbf{p}}\Big] f_\sigma(t,\mathbf{s},\mathbf{p}) =$$

$$e_\sigma\frac{1}{N_\sigma}\left(\mathbf{E}^\sigma(t,\mathbf{s}) + \tfrac{1}{c}\boldsymbol{v}_\sigma(\mathbf{p})\times\mathbf{B}^\sigma(t,\mathbf{s})\right)\cdot\nabla_{\mathbf{p}} f_\sigma(t,\mathbf{s},\mathbf{p}) + \mathcal{R}\left[f_\sigma\right](t,\mathbf{s},\mathbf{p}),$$

Vlasov-MBLTP results when r.h.s. is negligible!

The Einstein–Infeld–Hoffmann Legacy (cont.^{d})

On a smooth Lorentz manifold equipped with three times continuously differentiable metric, the twice contracted second Bianchi identity reads

$$\nabla_\mu G^{\mu\nu} = 0^\nu.$$

If that spacetime is a solution to Einstein's field equations,

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu},$$

then the second Bianchi identity implies the conservation laws

$$\nabla_\mu T^{\mu\nu} = 0^\nu$$

for the energy and momentum densities of the "matter" fields. But if "matter" includes point particles, then $G^{\mu\nu}$ is not differentiable at the singularities that represent the particle worldlines. Q: Is there a distributional version of the second Bianchi identity for spacetimes with timelike singularities?

A modest rigorous beginning ...

A. Burtscher, A.S. Tahvildar-Zadeh, and myself, in
"Weak second Bianchi identity for static, spherically symmetric spacetimes with timelike singularities,"
Class. Quantum Grav. **38** 185001, 31pp. (2021)
established what the title of our paper says, for charged timelike singularities in a class of electromagnetic vacua that includes the Born–Infeld vacuum, but NOT the BLTP vacuum.
Our result implies the consistency of **certain** static spherically symmetric spacetimes exhibiting a timelike charged singularity with a law of motion that tells the charge: "Don't move!"

Obviously, this is only a first baby step !

Fin!

MANY THANKS TO THE AUDIENCE FOR LISTENING!

MANY THANKS TO THE ORGANIZERS:
Angelo Bassi (Trieste)
Dirk - André Deckert (Munich)
Paula Reichert-Schürmer (Munich)
Ward Struyve (Leuven)
Fabian Nolte (Munich)

Abraham and Lorentz: $\delta_{\mathbf{q}(t)}(\mathbf{s}) \longrightarrow \frac{1}{|B_a|}\mathbf{1}_{B_a(\mathbf{q}(t))}(\mathbf{s})$

Since Lorentz electrodynamics with point charges is ill-defined, if one insists on the Maxwell–Lorentz field equations, then one needs to replace charged point particles with extended ones.
Replacing the point particle by a rigidly moving and spinning ball yields the Abraham model; insisting the spherical shape holds only in the particle's rest frame yields "the" Lorentz model.

RIGOROUS RESULTS for the ABRAHAM MODEL

G. Bauer and D. Dürr, "The Maxwell–Lorentz system of a rigid charge," Ann. Inst. H. Poincaré **2**, 179–196 (2001).
A.I. Komech and H. Spohn, "Long-time asymptotics for the coupled Maxwell–Lorentz Equations," Commun. PDE **25**, 559–584 (2000).
M. Kunze and H. Spohn, "Adiabatic limit of the coupled Maxwell–Lorentz Equations," Ann. Inst. H. Poincaré **1**, 625–653 (2000). (and many more follow-up publications)

ML and MBI and MBLTP Electromagnetic Field Theory

Rigorous Results on the Field Cauchy Problems

- ML field Cauchy problem (standard):
  Global well-posedness (weak) with "arbitrary" data.
- MBLTP field Cauchy problem (standard):
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- MBI field Cauchy problem:
  - Global well-posedness (classical) with small data (no charges!) (J. Speck; F. Pasqualotto)
  - Finite-time blow up with certain plane wave data (no charges!) (Y. Brenier; cf. D. Serre)
  - Existence and Uniqueness of static finite-energy solutions with N fixed point charges; real analyticity away from point charges (M.K.; cf. Bonheure et al.)

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