The Stochastic Evolution of Isolated Systems in Quantum Mechanics[1]

Dedicated to the memory of *Detlef Dürr* and *Vaughan Jones*

"Das andere große Problem, mit dem ich mich seit etwa 1900 befasst habe, ist das der Strahlungs- und Quantentheorie. . . . Den Rest meines Lebens werde ich wohl der grundsätzlichen Klärung dieses Problems widmen, wie gering auch die Aussichten auf ein Erreichen dieses Zieles erscheinen mögen." (Albert Einstein, 1924)

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Acknowledgements and Summary

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"It seems clear that the present quantum mechanics is not in its final form." (Paul Adrien Maurice Dirac)

Summary: This talk is about the *foundations of quantum theory*, a thorny subject haunted by controversy and confusion, not to make more critical comments. I will start by trying to describe some of the key problems and facts encountered in the foundations of QM.

Main purpose of talk: Survey of results of our search for a plausible **Completion of Quantum Mechanics**.

Summary - ctd.

"The constant element in physics, since Newton, is not a configuration or a geometrical form, but a **Law of Dynamics***."* (Werner Heisenberg)

I will focus on the derivation of a general *non-linear, stochastic Law of Dynamics* for *non-relativistic* Quantum Mechanics (QM) from basic principles (superseding Schrödinger evolution) and on some of the recent applications of this novel law.

Concretely: I will apply our general results to problems in quantum optics: The quantum-mechanical time evolution of physical systems consisting of charged matter interacting with the quantized electromagnetic field will be shown to be *non-linear*, *dissipative* and *stochastic*, featuring isolated random events sometimes called **"quantum jumps"**.

Explicit results are presented for an idealized description of such systems arising in a limiting **non-relativistic** regime where the *velocity of light*, c, tends to ∞. These equations involve a type of stochastic process, which I call **"Quantum Poisson Jump Process"**. I will introduce this process, present some of its mathematical properties and sketch applications to the theory of fluorescence of atoms.

1. General Introduction to Problems of QM

I take it for granted that the audience is somewhat familiar with **text-book QM**: Let S be a physical system to be described by QM; then

- *Pure states* of S = unit rays in a Hilbert space, $\mathcal{H}$, *mixed states* = density matrices acting on $\mathcal{H}$.
- *Physical quantities* characteristic of S are represented by certain selfadjoint operators acting on $\mathcal{H}$.
- *Time evolution* of states of S is described by a *Schrödinger-von Neumann equation* – **except** when a physical quantity of S is measured (!).
- *Effect of a measurement* of a physical quantity on the state of S is described by *von Neumann-Lüders postulates* : Transition to mixed state, followed by state-reduction / collapse + *Born's Rule*.

The Schrödinger-von Neumann equation for the evolution of states is **linear** and **deterministic**. The only information it contains concerns *spectral properties* of the *Hamiltonian*, H_S, generating time evol. – which is not much! Yet, experiments tell us that QM is fundamentally **probabilistic**. $\rightarrow$ **"Measurement Problem."**

Shortcomings of text-book QM, basic claims

This problem has motivated the *von Neumann-Lüders postulates*, which, however, cover up a lack of theory-intrinsic understanding of **measurements as physical processes** and lead to serious difficulties:
For example, it is posited that, *independently of the state in which S has been prepared,* measurements of *arbitrary physical quantities* of S can be carried out at *arbitrary times* and *arbitrary quickly* – which would lead to *infinitely large energy fluctuations (!)* – and *without including the measurement equipment in the quantum-mechanical description.* $\longrightarrow$ Fundamental role of *"observers."* – All this strikes me as quite **absurd**!

These defects of text-book QM are $\pm$ well known; but how to cure them remains very controversial $\longrightarrow$ problem of *"interpretations of QM"*: *different versions of Copenhagen, Everett, relational QM, Qbism, ... , Bohmian mechanics.* – I will sketch the **"*ETH*-Approach"** for you.

To begin with, the following claims will be substantiated on examples:

I. QM is intrinsically **probabilistic.** – Root: **EM field**.

II. Time evolution of **individual** systems **not** described by *Schrödinger-von Neumann equation* $\longrightarrow$ must search for **Law of Dynamics** in QM: *Non-linear stochastic dynamics* – Sect. 2, and Part II, Sect. 5.

Our tasks in attempting to complete QM

"Every experiment destroys some of the knowledge of the system which was obtained by previous experiments." (Werner Heisenberg)

III. To find a general **Law of Dynamics** in QM, we must introduce:

(i) appropriate notions of *"potentialities"* ("potential *events*") and of *"actualities"* ("(actual) events / facts");
(ii) a dichotomy between past and future (Aristotle) $\longrightarrow$ *"arrow of time"*; and a fundamental mechanism of *dissipation* (EM field);
(iii) an appropriate notion of *"states"*.

Remarks: Since, as *Heisenberg* intuited, the time evolution of states in QM distinguishes between past and future, the *Aristotelian dichotomy* between **"past"** (which consists of *actualities,* i.e., is **factual**) and **"future"** (which consists of **potentialities**) – separated by the **"present"**, in which certain (*which?*) potentialities *actualize* – must be properly incorporated into QM.

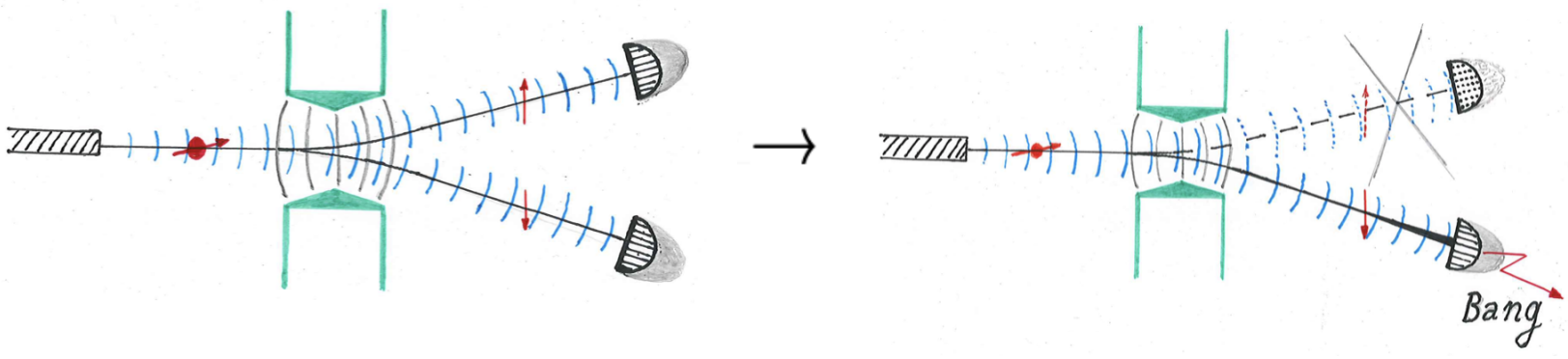
Physics is primarily a science enabling one to make predictions concerning the future. **States** should serve to make such predictions. Thus, states should be viewed as mathematical devices enabling one to make predictions about the likelihood of future potentialities to actualize.

QM is intrinsically probabilistic – item 1.

Much of the confusion still prevailing in QM is due to a lack of taking item III. (i) - (iii) properly into account.

I. The probabilistic nature of QM. Fact: In general one can only predict *probabilities* of different outcomes of experiments / measurements.

Example: Stern-Gerlach experiment with silver atoms (spin 1/2)



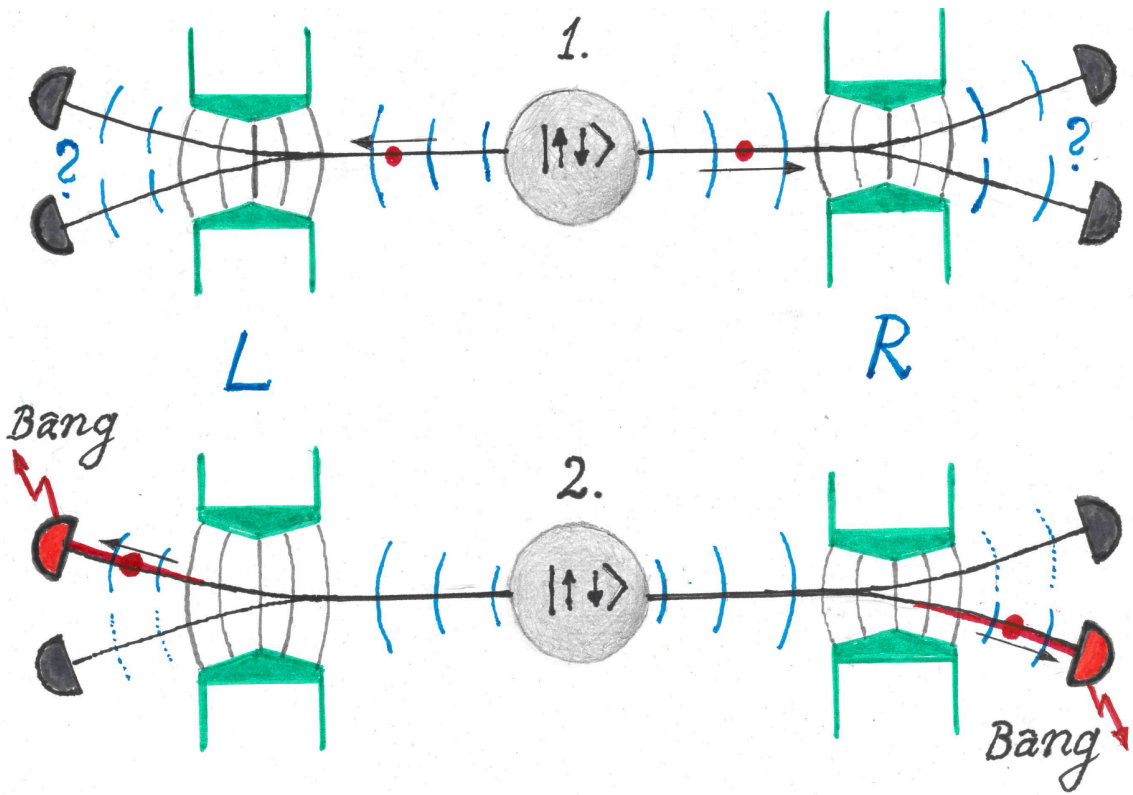
Schödinger evolution

or upper detector fires
true evolution

Only **probabilities** of the event that the upper / lower detector is firing are predicted by QM; and this has to be a basic feature of a fundamental law of dynamics.

Schrödinger equation does not describe propagation of states of *individual* systems – item II

II. Prepare a pair of silver atoms in an entangled *spin-singlet state* and orbital wave fus. such that one atom propagates to the left while the other atom propagates to the right. Then the Schrödinger equation predicts behavior 1. (below), while all experiments reveal **events** as in 2. (or 2., with upper and lower detectors interchanged on the left *and* on the right), exhibiting **correlations** between left and right:



"Non-locality" of QM versus "Einstein causality"

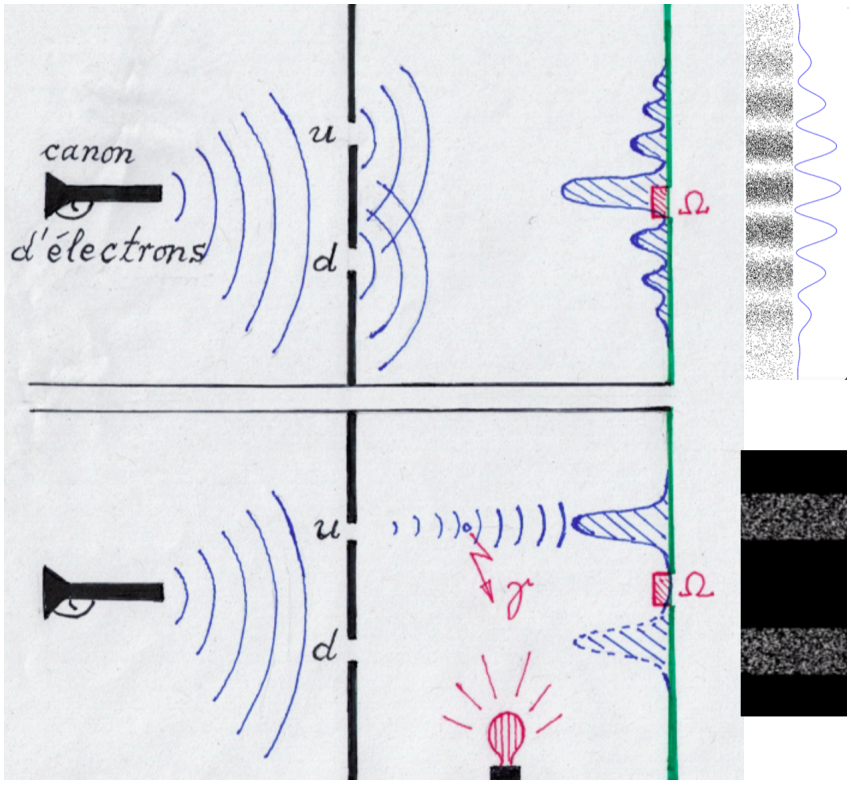
If the Schrödinger equation described the evolution of states of individual systems then spin measurements on the left would **not** bias the outcome of spin measurements on right $\longrightarrow$ item II ! That there **is** such a bias is interpreted as *"non-locality"* of QM.
This analysis has been presented in *Faupin-F-Schubnel* (2015).

Another Gedanken-Experiment leading to the conclusions in item II is *Wigner's friend paradox*; see *Wigner, Hardy, Frauchiger-Renner* (2018)...

"Non-locality" of QM and *Einstein causality* ($\nearrow$ RQFT): Measurements of components of spins of silver atoms can be made in *space-like separated regions* of space-time. Then the order in which these two measurements occur depends on the rest frame of the observer who records measurement data. For predictions of measurement data of different observers to be *compatible* the *operators representing the results of these measurements must commute,* as assumed in RQFT!

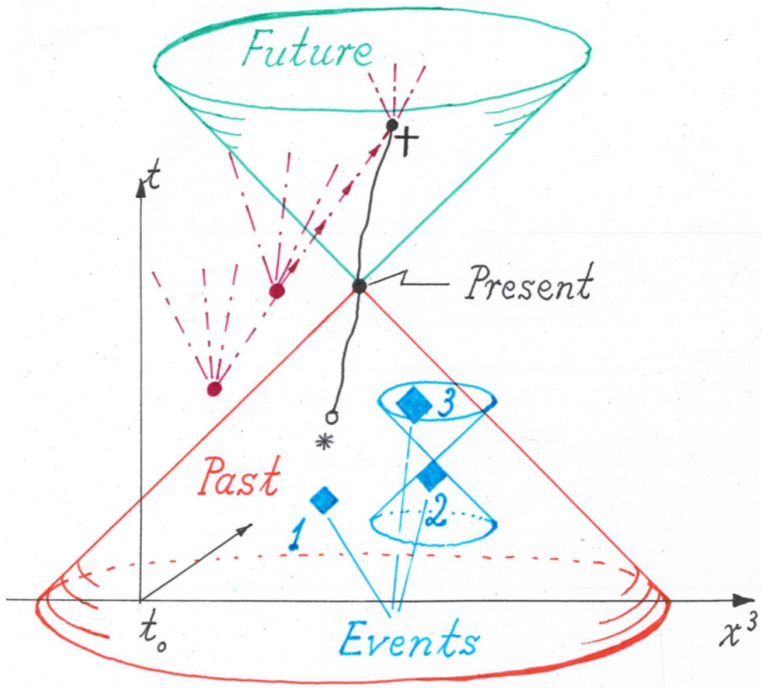
The *"non-locality of QM"* is often mis-represented, and the talk about a tension between QM and Relativity Theory is misguided.

The importance of a clear notion of *"events"* – item III. (i)



In the upper situation, the question whether the particle went through the **u**-*slit* or through the **d**-*slit* before hitting the screen is *meaningless*; for, it is a question that does **not** concern an *"event"* that can ever happen. In QM, the picture of a particle being point-like and always having a precise position is *wrong* ! – The lower picture is about a measurement, to the right of the double slit, of the *approximate position* of the particle by light scattering $\longrightarrow$ interference pattern disappears.

Dichotomy between past and future and the notion of states – item III. (ii) & (iii)



potentialities

facts

Past = History of Events (*facts*) / Future = Ensemble of *potentialities*

This dichotomy must be repr. in QM → fundamental arrow of time!

The past is factual. – The role of **states** is to serve to predict the likelihood of **potential events** (*potentialities*) actualizing in the future.

2. Sketch of the *ETH*-Approach to QM

In QM, the time evolution of *"ensemble states"*, i.e., of *averages of states* taken over a *large ensemble* of identical, identically prepared systems, is usually taken to be *linear* and *deterministic* (Schrödinger-von Neumann –, or Lindblad-type evolution). However, the quantum-mechanical time evolution of the *state of an* **individual** *system* is *non-linear* and *stochastic*.

A conceptual explanation of this fundamental fact is the core of the **"*ETH*-Approach to QM"**, a special case of which appropriate for a precise formulation of **non-relativistic QM** will be sketched presently. It is based on the following **four ingredients**:

A. **Physical quantities** of a system, S, are represented by abstract, selfadjoint linear ops. $\widehat{X}, \widehat{Y}, \ldots$ belonging to a family, $\mathcal{O}_S$. At every time t, all the elements of $\mathcal{O}_S$ are represented by concrete operators, $\mathcal{O}_S \ni \widehat{X} \mapsto X(t) = X(t)^* \in B(\mathcal{H})$, acting on a Hilbert space $\mathcal{H}$. The t-dependence of the operators $X(t)$ is given by the well known **Heisenberg equations**: $\dot{X}(t) = i\big[H_S, X(t)\big]$, $(\hbar = 1)$.

B. **Algebras of "potential events"** are defined as

$$\mathcal{E}_{\geq t} := \Big\langle X(t') \,\Big|\, X(t') \text{ rep. a phys. quantity } \widehat{X} \in \mathcal{O}_S,\ t' \geq t \Big\rangle^{-}, \quad (1)$$

Ingredients of the *ETH*-Approach to QM

It follows immediately from the definition that

$$\mathcal{E}_{\geq t'} \subseteq \mathcal{E}_{\geq t}\,, \qquad \text{whenever} \quad t' > t\,,$$

and the Heisenberg equations imply that

$$\mathcal{E}_{\geq t'} = e^{i(t'-t)H_S}\,\mathcal{E}_{\geq t}\,e^{i(t-t')H_S}\,, \quad \text{for arbitrary} \quad t, t',$$

where H_S is the Hamiltonian of S (assumed autonomous); i.e.,

$$\mathcal{E}_{\geq t} \simeq \mathcal{E}_{\geq 0}, \quad \forall t.$$

C. **Potential events** (potentialities), $\mathfrak{e}$, in QM possibly happening at some time $\geq t$ are given by *partitions of unity by disjoint orthogonal projections* in the algebra $\mathcal{E}_{\geq t}$

$$\mathfrak{e} = \{\pi_\xi \,|\, \xi \in \mathfrak{X}\} \subset \mathcal{E}_{\geq t}, \quad \mathfrak{X} \text{ an index set.}$$

D. **States** of a system $\simeq S$ **at time** t are defined to be *normalized, positive, linear functionals on the algebra* $\mathcal{E}_{\geq t}$.

If and only if $\mathcal{E}_{\geq t}$ is *independent* of t (as for *closed* systems and relativistic systems with a strictly positive mass gap) then, in the *Heisenberg picture,* states are independent of t.

Postulates for Non-Relativistic Quantum Mechanics

P-1) "Principle of Diminishing / Declining Potentialities"

An isolated but open system S is characterized by the property

$$\mathcal{E}_{\geq t'} \subsetneqq \mathcal{E}_{\geq t}\,, \quad \forall\, t' > t\,. \qquad (PDP) \qquad (2)$$

In QED, (PDP) is a consequence of *Huygens' Principle*; (photons are *massless*). – If (PDP) holds then states generally depend *non-trivially* on time t; (a consequence of *entanglement*).

P-2) "Type-I property"

The algebra $\mathcal{E}_{\geq 0}$ is isomorphic to the algebra of bounded operators on a Hilbert space $\mathcal{H}_0 \subseteq \mathcal{H}$. – Note: **P-2)** is **special** to NR QM!

P-2) implies that a *state* on $\mathcal{E}_{\geq 0}$ is given by a **density matrix**, ρ_0, on $\mathcal{H}_0$. The isomorphism $\mathcal{E}_{\geq t} \simeq \mathcal{E}_{\geq 0}$ implies that a state of S at time t is given by a density matrix on $\mathcal{H}_0$, too. – Let $\mathfrak{E}_S$ be a large ensemble of systems $\simeq S$; let ρ_0 be the state of all the systems in $\mathfrak{E}_S$ at an initial time $t_0 = 0$. Then the average, ρ_t, over $\mathfrak{E}_S$ of their states at a later time $t > 0$ (which is called an **"ensemble state"**) is given by **restricting** the initial state ρ_0 to the algebra $\mathcal{E}_{\geq t}$. In the Schrödinger picture, this means that

$$\mathrm{Tr}\big(\rho_t \cdot X\big) := \mathrm{Tr}\big(\rho_0 \cdot e^{itH_S} X e^{-itH_S}\big), \quad \forall\, X \in \mathcal{E}_{\geq 0}\,. \qquad (3)$$

The state reduction postulate

When formulated with precision, Equation (3) implies that

$$\rho_t = e^{t\mathcal{L}}[\rho_0], \qquad t > 0, \tag{4}$$

where $\left\{ e^{t\mathcal{L}} \,\middle|\, t \geq 0 \right\}$ is a *semigroup of completely positive linear maps* (i.e., $\mathcal{L}$ is a Lindblad generator) gen.mapping *pure* to *mixed* states.

P-3) **"State-Reduction Postulate"**

At all times $t > 0$, the state of an **individual** *system in $\mathfrak{E}_S$ satisfying Postulates P-1) and P-2) is given by a finite-rank orthogonal projection, π_t, belonging to a potential event $\mathfrak{e} \subset \mathcal{E}_{\geq t}$ that actualizes at time t; π_t is a random object obtained by* **"purifying" / "unraveling"** *the evolution described in Eq. (4). A generalized* **Born Rule** *holds.*

The statements in **P-3)** will be made precise in Sect. 5.

The **new elements** of the *ETH-***Approach to QM**, as compared to text-book QM, are ingredient D. (notion of **states**), and Postulates **P-1)** (*PDP*) and **P-3)** (state-reduction postulate) – with far-reaching consequences (!), e.g., for solving the infamous *"measurement problem"* in QM (a "no-problem"), the *"information paradox,"* etc..

3. Non-relativistic models of matter interacting with the quantized radiation field – fluorescence of atoms

We will consider systems, S, consisting of a *static atom* (orbital motion neglected) with $N = 2, 3, \ldots$ internal states of energies $E_0 < \cdots < E_{N-1}$, described by a complete orthonormal system, $\{\psi_0, \ldots, \psi_{N-1}\}$, of eigenstates of a matrix H_A, the *Hamiltonian* of the atom, acting on the Hilbert space $\mathfrak{h}_A := \mathbb{C}^N$:

$$H_A\psi_j = E_j\psi_j, \qquad \text{for } \; j = 0, 1, \ldots, N-1\,.$$

The atom is coupled to the quantized electromagnetic field. Given (ij), one specifies a *transition amplitude*, d_{ij}, for a transiton from ψ_i to ψ_j, $i, j = 0, 1, \ldots, N-1$, with $d_{ij} = \overline{d_{ji}} \in \mathbb{C}$, accompanied by *emission* of a photon of frequency $\omega_{ij} \approx \hbar^{-1}[E_i - E_j]$ if $i > j$, and by *absorption* of a photon of frequency $\omega_{ij} \approx \hbar^{-1}[E_j - E_i]$ if $i < j$.

The state space of the photons is the usual Fock space, $\mathcal{F}$.

In text books, the dynamics of the atom coupled to the quantized radiation field is described by unitary evolution on the Hilbert space $\mathcal{H} := \mathfrak{h}_A \otimes \mathcal{F}$ generated by a *Hamiltonian*

Dynamics of atom coupled to the radiation field

$$H := H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_f + eH_I(\{d_{ij}\}),$$

where H_f is the Hamiltonian of the non-interacting electromagnetic field, e is the elementary electric charge, and H_I is an operator determined by the transition amplitudes $\{d_{ij}\}$ and describing the creation or annihilation of a photon.

In a **non-relativistic regime** where the velocity of light $c \to \infty$, this model fits perfectly into the formalism of the *ETH*-Approach (Sect. 2): The algebras $\mathcal{E}_{\geq t}$ (see Eq. (1)) are isomorphic to algebras $B(\mathfrak{h}_A \otimes \mathcal{F}_{>t})$, where $\mathcal{F}_{>t}$ is the subspace of $\mathcal{F}$ obtained by applying arbitrary electromagnetic field operators localized at times $\geq t$ to the vacuum. It is not difficult to show that the **Principle of Diminishing Potentialities** (see P-1), Eq. (2)) holds.

For simplicity we suppose that, at an initial time $t_0 = 0$, S is prepared in a state without any photons; i.e., the radiation field is in its so-called **vacuum state**, $|\emptyset\rangle$. In the limit where $c \to \infty$, the state of S restricted to the algebra $\mathcal{E}_{\geq t}$ is then always given by the tensor product of an atomic density matrix Ω_t with the projection, $|\emptyset\rangle\langle\emptyset|$, onto the vacuum state of the radiation field, **for all** times t.

The time evolution of ensemble states

This is because if $c = \infty$ photons emitted at times $< t$ have already escaped to spatial infinity at time t, This fact allows us to *entirely neglect the degrees of freedom of the radiation field, provided the latter is prepared in its vacuum state.*

Let Ω_0 be the density matrix describing the initial state of the atom at time $t_0 = 0$. Then the *ensemble state* of S at time t, obtained by restricting the initial state $\rho_0 := \Omega_0 \otimes |\emptyset\rangle\langle\emptyset|$ to the algebra $\mathcal{E}_{\geq t}$, is given by

$$\rho_t = \Omega_t \otimes |\emptyset\rangle\langle\emptyset| ,$$

where Ω_t is a certain density matrix on $\mathfrak{h}_A$. As asserted in Sect. 2, the *evolution equation* for Ω_t is given by a **Lindblad equation**

$$\dot{\Omega}_t = \mathfrak{L}_\alpha[\Omega_t] , \quad \text{with}$$
$$\mathfrak{L}_\alpha[\Omega] := -i\hbar^{-1}\big[H_A, \Omega\big] + \alpha \sum_k \Big[V_k \Omega V_k^* - \frac{1}{2}\big\{\Omega, V_k^* V_k\big\}\Big] , \tag{5}$$

where $\alpha = e^2$ $(\hbar = 1) \rightarrow$ *dissipative nature of evolution of atom is entirely due to its coupling to the radiation field!*

The time evolution of states of individual systems?

The operators V_k, $k = 1, 2, \ldots$, on $\mathfrak{h}_A$ can be calculated from the transition amplitudes $\left\{ d_{ij} \,|\, i, j = 0, 1, \ldots, N-1 \right\}$ and the "form factors" of photon creation- and annihilation operators appearing in the interaction Hamiltonian H_I.

"Purifying" / "unraveling" of Eq. (5) will yield the non-linear **Law of the stochastic dynamics of individual quantum systems** $\simeq S$, which is given by what we call a **"quantum Poisson jump process"** (see Sect. 5). **"Unraveling"** involves interpreting the spectral projections of a density matrix describing an ensemble state at time t as an *"event" actualizing at time* t.

In the next Section, I explain the process of "unraveling" a linear evolution equation for ensemble states on the simpler example of the discrete diffusion equation "unravelled" by the law of simple random walks on a lattice.

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4. Diffusion and the theory of random walkers

We consider a large ensemble, $\mathfrak{E}$, of *identical, non-interacting random walkers* on the simple lattice $\mathbb{Z}^\nu, \nu = 1, 2, 3, \ldots$ An *"ensemble state"* at time t is given by the *density*, ρ_t, of random walkers on $\mathbb{Z}^\nu$. The time-dependence of ρ_t is governed by the *diffusion equation*

$$\dot{\rho}_t(x) \equiv \partial\rho_t/\partial t = D\big(\Delta\rho_t\big)(x) = D\Big[\sum_{y:|y-x|=1} \rho_t(y)\Big] - 2\nu D\,\rho_t(x)\,, \quad (6)$$

where Δ is the discrete Laplacian, $D =$ diffusion constant, and the sum on the right side of (6) extends over all sites y that are nearest neighbors of x (indicated by $|y - x| = 1$).
The diffusion equation is a **linear, deterministic evolution equation** for the ensemble state ρ_t; it is the *analogue* of the Lindblad equation (5).

A solution ρ_t of (6) is non-negative, for all times $t > 0$, provided ρ_0 is non-negative; (a consequence of the fact that the heat kernel, $\big(e^{tD\Delta}\big)_{xy}$, is positivity preserving). Moreover, $\sum_{x\in\mathbb{Z}^\nu} \rho_t(x)$ is *independent* of time t; hence we may (re-)normalize ρ_t such that $\sum_x \rho_t(x) = 1$, for all $t \geq 0$.

The stochastic motion of individual random walkers

We would like to understand what kind of stochastic motion of a single random walker $\omega \in \mathfrak{E}$ implies that the time-dependence of an ensemble state ρ_t is determined by the diffusion equation (6). For this purpose, we have to *"unravel"* this equation.

It helps intuition to re-write it in the form

$$\rho_{t+dt}(x) = \rho_t(x) + D\big(\Delta\rho_t\big)(x) \cdot dt + \mathcal{O}(dt^2)\,, \tag{7}$$

where dt is assumed to approach 0.

"Ontology":

The "true" state of an individual random walker $\omega \in \mathfrak{E}$ at an arbitrary time t is a **site** $x_\omega(t) \in \mathbb{Z}^\nu$, corresp. to $\rho_t(x;\omega) := \delta_{x_\omega(t)}(x)$, $x \in \mathbb{Z}^\nu$, (a *"pure state"*), where $\delta_y(x) \equiv \delta_{yx}$ is the Kronecker δ.

During a time interval $[t, t+dt)$, a random walker ω may remain at $x_\omega(t)$, or it may jump to a nearest-neighbor site y, with $\big|y - x_\omega(t)\big| = 1$; i.e., its state may "jump" from $\delta_{x_\omega(t)}$ to δ_y, with $\big|y - x_\omega(t)\big| = 1$. Equation (7) determines the *probabilities* for ω to remain at $x_{\omega(t)}$ or to jump to site y, with $\big|y - x_\omega(t)\big| = 1$.

The stochastic motion of an individual random walker

We write (7) as

$$\rho_{t+dt}(x) = \big[1 - 2D\,\nu\,dt\big]\rho_t(x) + \sum_{y:|y-x|=1} \big[D\,dt\big]\rho_t(y) + \mathcal{O}(dt^2)\,,$$

with $\rho_t(x) = \rho_t(x;\omega) = \delta_{x_\omega(t)}(x) \Rightarrow \rho_{t+dt}$ is a *convex superposition of pure states* corresponding to sites $x_\omega(t)$ and $x_\omega(t)+\delta$, where δ ranges over lattice unit vectors. The coefficients appearing in this superposition can be interpreted as *probabilities.* If dt is tiny, $\rho_{t+dt}(\cdot;\omega)$ is given by

$$\rho_{t+dt}(\cdot;\omega) = \begin{cases} \delta_{x_\omega(t)}, & \text{with prob. } p_{nj}[t,t+dt] := 1 - 2\nu D\,dt\,, \\ \delta_{x_\omega(t)+\delta}, & \text{with prob. } p^\delta[t,t+dt] := D\,dt,\ \forall\delta, \end{cases} \quad (8)$$

where "*nj*" stands for "no jump", "δ" stands for a jump from $x_\omega(t)$ to $x_\omega(t)+\delta$ in the time interval $[t,t+dt)$; (multiple jumps have negligible probability $o(dt)$). Note that $p_{nj}[t,t+dt] + \sum_\delta p^\delta[t,t+dt] = 1$.

The first equation in (8) implies that

$$p_{nj}[t',t''] = e^{-2\nu D(t''-t')}, \quad \text{for } t'' > t'.$$

A Poisson jump process

Suppose $x_\omega(\tau)$ is a random walker starting at $\tau = 0$ at an arbitrary site $\omega(0) \in \mathbb{Z}^\nu$ and making $n = \ell(\omega)$ jumps along a given SRW ω, at times $\tau \in [t_k, t_k + dt_k)$, $k = 1, \ldots, n$, until it stops at some time t. By (8), the probability that a random walker traces out this "history" is given by

$$W_\omega\big[t_1, \ldots, t_n\big] \prod_{j=1}^{n} dt_j := $$

$$= \Big\{ \prod_{k=1}^{n} p_{nj}[t_{k-1}, t_k]\, p_{\omega(k)-\omega(k-1)}[t_k, t_k + dt_k] \Big\} p_{nj}[t_n, t]$$

$$= e^{-2\nu D \cdot t}\, dt_1 \ldots dt_n\,, \quad \text{where} \quad t_0 = 0 < t_1 < \cdots < t_n < t\,.$$

Poisson jump process

$\longrightarrow$ Mean square distance travelled by random walker in time t is $\propto t$:

$$\mathbb{E}_\omega\big[x_\omega(t) - x_\omega(0)\big]^2 = \sum_n \underbrace{\mathbb{E}_\omega\big[\omega(n) - \omega(0)\big]^2}_{=n} \frac{(2\nu D \cdot t)^n}{n!} e^{-2\nu D \cdot t}$$

$$= 2\nu D \cdot t \;=\; \sum_{y \in \mathbb{Z}^\nu} \big(e^{tD\Delta}\big)_{xy} \big[y - x\big]^2; \quad \text{etc.}$$

5. A theory of fluorescence of atoms derived from the *ETH*-Approach

We consider a system $\simeq S$, as in Sect. 3, prepared at time $t_0 = 0$ in an initial state $\Omega_0 \otimes \big|\emptyset\big\rangle\big\langle\emptyset\big|$, where

$\Omega_0 := \Pi_0$ is a **pure** state, i.e., Π_0 is a rank-1 orthogonal projection,

and $\big|\emptyset\big\rangle\big\langle\emptyset\big|$ is the **vacuum state** of the radiation field ($\nexists$ any photons), which we describe in the limiting regime where $c \to \infty$. As in Sect. 3, we suppose that $\nexists$ any detectors recording photons.

I recall that, in the Schrödinger picture, the *ensemble state* at time t is then given by $\Omega_t \otimes \big|\emptyset\big\rangle\big\langle\emptyset\big|$, $\forall\, t$, where Ω_t satisfies a *Lindblad equation*

$$\Omega_{t+dt} = \Omega_t + \mathfrak{L}_\alpha[\Omega_t]dt + \mathcal{O}(dt^2)\,, \quad \text{with}$$

$$\mathfrak{L}_\alpha[\Omega] := -i\big[H_A, \Omega\big] + \alpha \sum_k \big[V_k \Omega V_k^* - \frac{1}{2}\big\{\Omega, V_k^* V_k\big\}\big]\,, \qquad (9)$$

$$\Omega_{t=0} = \Omega_0\,.$$

According to the *ETH-Approach*, the state of an **individual** system isomorphic to S is $\propto \Pi_t \otimes \big|\emptyset\big\rangle\big\langle\emptyset\big|$, where Π_t is an orthogonal projection of finite rank, $\forall$ times $t > 0$; (State Reduction Postulate, see Sect. 2, P-3)).

Diagonalizing Ω_{t+dt}

For simplicity, we assume that Π_t is of rank 1, i.e., a pure state. Then the state at time $t+dt$, when averaged over a large ensemble, $\mathfrak{E}_S$, of systems, **all** identical to S and prepared in the pure state $\Pi_t \otimes |\emptyset\rangle\langle\emptyset|$ at time t, is given by $\Omega_{t+dt} \otimes |\emptyset\rangle\langle\emptyset|$, where, according to Eq. (9),

$$\Omega_{t+dt} = \Pi_t + \mathfrak{L}_\alpha[\Pi_t]\, dt + \mathcal{O}(dt^2)\,. \tag{10}$$

The state Ω_{t+dt} isn't pure, anymore. Since $\Omega_{t+dt} = \Omega^*_{t+dt} > 0$, with $\mathrm{tr}\big(\Omega_{t+dt}\big) = \mathrm{tr}\,\Pi_t = 1$, the decomposition of Ω_{t+dt} into a *convex combination of disjoint projections* takes the form (↗ spectral theorem)

$$\Omega_{t+dt} = p_{nj}[t, t+dt]\,\widehat{\Pi}^0_{t+dt} + \sum_{\delta=1,\ldots,N-1} p^\delta[t, t+dt]\,\widehat{\Pi}^\delta_{t+dt}\,, \tag{11}$$

where $\widehat{\Pi}^\delta_{t+dt} = \frac{\Pi^\delta_{t+dt}}{\mathrm{tr}(\Pi^\delta_{t+dt})}$, the ev's $p_{nj} \equiv p^0$, $p^\delta, \delta \geq 1$, have the properties

$$p_{nj}[t, t+dt] = 1 - \mathcal{O}(dt) > 0,$$

$$p_{nj} > p^1 > \cdots > p^{N-1} > 0, \quad \text{with} \quad p^\delta = \mathcal{O}(dt),\ \forall\, \delta \geq 1\,,$$

$$p_{nj}[t, t+dt] + \sum_{\delta=1,\ldots,N-1} p^\delta[t, t+dt] = 1\,, \quad (nj = \text{“no jump”})\,.$$

The state-reduction postulate of the *ETH* - Approach

According to the **State Reduction Postulate**, **one** of the ops. $\widehat{\Pi}^{\delta}_{t+dt}$, $\delta = 0, 1, \ldots, N-1$, *randomly chosen,* is the state of an **individual** atom at time $t + dt$: According to the **Born Rule**, the probability, or frequency, that $\widehat{\Pi}^{\delta}_{t+dt}$ (for arb. δ) is chosen is given by $p^{\delta}[t, t+dt]$.

In order to come up with explicit expressions for these quantities, we apply **"infinitesimal (analytic) perturbation theory"** (*IPT*), considering Π_t in (10) as an *unperturbed operator* (H_0), $\mathfrak{L}_{\alpha}[\Pi_t]$ as a *perturbation* (V), and $dt =$ *"coupling const."* (strength of perturbation).

Digression on *IPT*: Let H_0 be an operator on $\mathbb{C}^N$ with a simple eigenvalue E_0 separated from the rest of its spectrum by a strictly positive gap. We let $\Pi := |\psi_0\rangle\langle\psi_0|$ be the projection onto the eigenvector, ψ_0, of H_0 corresponding to the eigenvalue E_0. Let V be an operator on $\mathbb{C}^N$, and consider the perturbed operator

$$H(\varepsilon) := H_0 + \varepsilon\, V, \qquad \text{with} \quad \varepsilon \equiv dt \ll 1\,.$$

We are interested in formulae for the ev $E_0(\varepsilon)$ of $H(\varepsilon)$ growing out of the unperturbed ev E_0 and the eigenprojection, $\Pi(\varepsilon)$, onto the eigenvector of $H(\varepsilon)$ corresp. to $E_0(\varepsilon)$.

Infinitesimal (analytic) perturbation theory

Setting $\Pi^{\perp} := \mathbf{1} - \Pi$, an anti-symmetric operator S is defined by

$$S := \big(H_0 - E_0\big)^{-1}\Pi^{\perp}\cdot V\cdot\Pi + \Pi\cdot V\cdot\Pi^{\perp}\big(E_0 - H_0\big)^{-1}. \tag{12}$$

IPT: *The ev $E_0(\varepsilon)$ and the eigenprojection $\Pi(\varepsilon)$ are given by*

$$\begin{aligned} E_0(\varepsilon) &= E_0 + \varepsilon\cdot\mathrm{tr}\big(\Pi\cdot V\big) + \mathcal{O}(\varepsilon^2)\\ \Pi(\varepsilon) &= \Pi - \varepsilon\big[S,\Pi\big] + \mathcal{O}(\varepsilon^2)\,. \end{aligned} \tag{13}$$

Up to errors of order $\mathcal{O}(\varepsilon^2)$, the remaining eigenvalues of $H(\varepsilon)$ can be found by diagonalizing the matrix $\Pi^{\perp}\cdot H(\varepsilon)\cdot\Pi^{\perp}$.

The equation for $E(\varepsilon)$ is called *Feynman-Hellmann theorem*, the one for $\Pi(\varepsilon)$ is *first-order (Rayleigh-Schrödinger) perturbation theory.*

Remark: Given a differentiable family, $\big\{H_t\,\big|\,0\le t\le 1\big\}$, of operators with the property that the spectrum and the eigenvectors of H_0 are known explicitly, one can find the spectrum and the eigenvectors of $H_t, 0 < t \le 1$, by solving an *initial value problem consisting of a system of ordinary differential equations* derived from formula (13). This *very useful* variant of perturbation theory is what I call *IPT*. (Applications!)

IPT and the *stochastic, non-linear* evolution of *individual* systems in QM

We use *IPT* to derive from the Lindblad eq. (10) a system of ODE's for the states $\widehat{\Pi}_t^0 = \Pi_t$ in the absence of quantum jumps, see (11), assuming for simplicity they are **pure** (i.e., rank 1), at all times t. We set $H_0 := \Pi_t$, $V := \mathfrak{L}_\alpha[\Pi_t]$, $\varepsilon := dt$. Then spec(H_0) is given by $\{1, \underbrace{0, \ldots, 0}_{N-1 \text{ times}}\}$.

$$Eq.\,(12) \;\Rightarrow\; S \equiv S_t := -\Pi_t^\perp \cdot \mathfrak{L}_\alpha[\Pi_t] \cdot \Pi_t + \Pi_t \cdot \mathfrak{L}_\alpha[\Pi_t] \cdot \Pi_t^\perp\,, \qquad (14)$$

which is a well-defined anti-symmetric operator on $\mathbb{C}^N$. Applying *IPT* we find the following system of differential equations:

(i) Feynman-Hellmann $\Rightarrow$

$$p_{nj}[t, t+dt] = 1 + \mathrm{tr}\big(\Pi_t \cdot \mathfrak{L}_\alpha[\Pi_t]\big)\, dt + \mathcal{O}(dt^2), \text{ hence}$$

$$\frac{\ell \mathrm{n}\, p_{nj}[t, t+dt]}{dt} = \mathrm{tr}\big(\Pi_t \cdot \mathfrak{L}_\alpha[\Pi_t]\big) < 0 \;\Rightarrow$$

$$\boxed{p_{nj}[0, t] = \exp\Big\{ \int_0^t \mathrm{tr}\big(\Pi_s \cdot \mathfrak{L}_\alpha[\Pi_s]\big) ds \Big\} < 1}$$

Time-evolution of state in absence of "quantum jumps"

(ii) Time-dependence of state, Π_t, in the *absence* of quantum jumps, (i.e., eigenprojections corresp. to ev $p_{nj}[t, t+dt]$ are chosen in an open interval of times containing t): Eqs. (11) & (13) $\Rightarrow$

$$\boxed{\frac{d\Pi_t}{dt} = \Pi_t^{\perp} \cdot \mathfrak{L}_\alpha[\Pi_t] \cdot \Pi_t + h.c.}$$

This is a system of **non-linear** (cubic) differential eqs. for Π_t.

(iii) "Quantum jumps": The **spectrum** of the **non-negative** matrix $\Pi_t^{\perp} \cdot \mathfrak{L}_\alpha[\Pi_t] \cdot \Pi_t^{\perp}$ is given by $\left\{ \frac{p^\delta[t,t+dt]}{dt} \middle| \delta = 1, 2, \dots \right\}$. Hence

$$0 < p^\delta[t, t+dt] = \mathcal{O}(dt), \ \delta = 1, 2, \dots, \quad \text{and}$$

$$p_{nj}[t, t+dt] + \sum_{\delta=1,2,\dots} p^\delta[t, t+dt] = 1.$$

(iv) For $\alpha = 0$ (i.e., atom decoupled from radiation field), one finds that

$$p_{nj}[0, t] \equiv 1 \quad \leftrightarrow \text{ no quantum jumps (!)}, \quad \frac{d\Pi_t}{dt} = -i\big[H_A, \Pi_t\big],$$

$\rightarrow$ *unitary evolution,* as expected!

State trajectories with "quantum jumps"

We now introduce an analogue of the *Wiener measure* of BM (more precisely of the measure defined at the end in Sect. 4). We suppose the state of a system $\simeq S$ exhibits "quantum jumps" at $t_1 < \cdots < t_n < t_{fin}$, with $t_1 > 0, t_{n+1} \equiv t_{fin}$. The state evolves continuously between jumps at times t_j and t_{j+1} according to the differential equations derived in item (ii); it is denoted by $\Pi_t^{\delta_j|\delta_{j-1}}$, with $t_j < t < t_{j+1}$, where $t_0 = 0$, $\Pi_{t=0}^{\delta_0|\delta_{-1}} = \Pi_0$ is the initial state of the system. We propose to calculate the probability of a trajectory, $\mathfrak{T}_n$, of states of an individual system $\simeq S$ given by

$$\mathfrak{T}_n := \left\{ \Pi_t^{\delta_j|\delta_{j-1}} \middle| t_j < t < t_{j+1}, j = 0, 1, \ldots, n \right\}, \tag{15}$$

where $\Pi_{t_j}^{\delta_j|\delta_{j-1}}$ is $(\propto)$ the eigenprojection corresponding to the eigenvalue $p^{\delta_j}[t_j, t_j + dt_j]$ of the density matrix

$$\Omega_{t_j+dt_j} = \Pi_{t_j}^{\delta_{j-1}|\delta_{j-2}} + \mathfrak{L}_\alpha[\Pi_{t_j}^{\delta_{j-1}|\delta_{j-2}}]\, dt_j\,,$$

see Eqs. (10) & (11). Following (i), we define

$$p_{nj}^{\delta_j|\delta_{j-1}}[t_j, t_{j+1}] := \exp\left\{ \int_{t_j}^{t_{j+1}} \mathrm{tr}\left(\Pi_t^{\delta_j|\delta_{j-1}} \cdot \mathfrak{L}_\alpha\left[\Pi_t^{\delta_j|\delta_{j-1}}\right] \right) dt \right\}.$$

A quantum-mechanical analogue of the Wiener measure

The probability, $W_{\mathfrak{T}_n}$, of the trajectory $\mathfrak{T}_n$ introduced in Eq. (15) is then given by

$$W_{\mathfrak{T}_n}\left[\delta_1, t_1, \ldots, \delta_n, t_n\right] \prod_{j=1}^{n} dt_j :=$$

$$= \left\{ \prod_{j=0}^{n-1} p_{nj}^{\delta_j|\delta_{j-1}}[t_j, t_{j+1}] p^{\delta_{j+1}}[t_{j+1}, t_{j+1} + dt_{j+1}] \right\} p_{nj}^{\delta_n, \delta_{n-1}}[t_n, t_{fin}] \,. \tag{16}$$

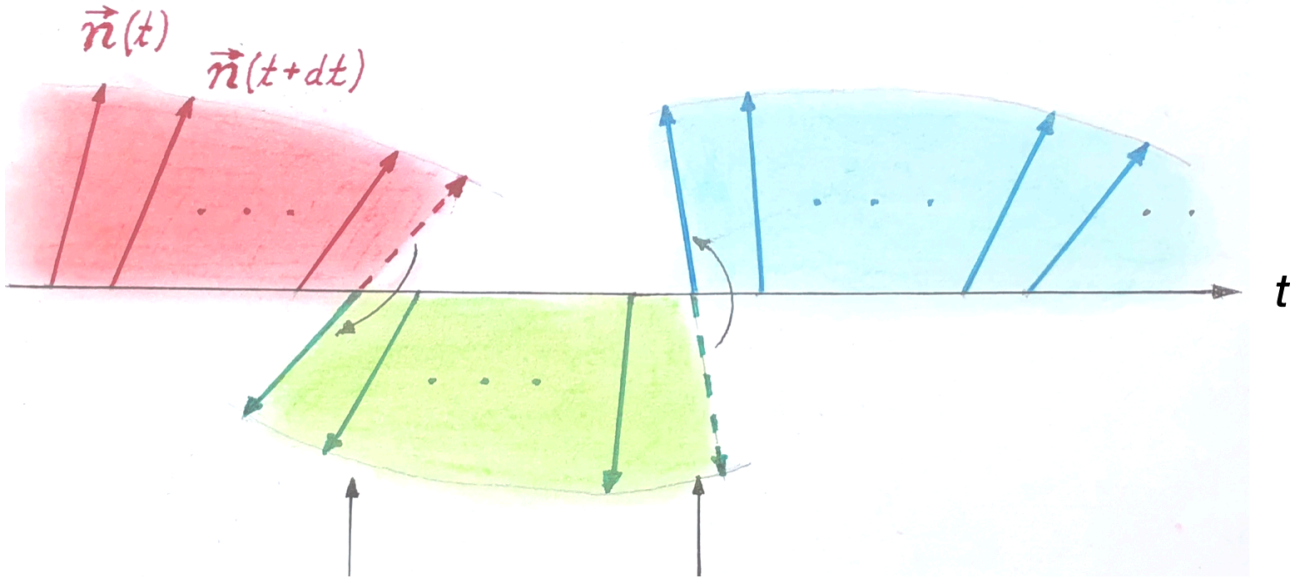
This formula serves to determine the probability of "measureable sets" of quantum trajectories, $\mathfrak{T}$, of **individual** systems $\simeq S$ with an arbitrary number of "quantum jumps" in the time interval $[0, t_{fin}]$.

Remarks: (1) If an average, $\mathbb{E}$, over the states in the trajectories $\mathfrak{T}_n$ is taken, using the "measures" $W_{\mathfrak{T}_n}$, with $n = 0, 1, 2, \ldots$, then we recover the ensembles states Ω_t obeying the Lindblad equation (10).

(2) The equation in (ii) can be re-written as a non-linear Schrödinger equation for a pure state vector Ψ_t, with $\Pi_t = \left|\Psi_t\right\rangle\left\langle\Psi_t\right| \equiv \Psi_t \cdot \Psi_t^*$.
$\rightarrow$ Novel approach to *"quantum chaos."*

6. The example of a two-level atom – fluorescence

A pictorial representation of quantum trajectories of states (denoted here by $\vec{n}_t$) of **individual** systems $\simeq S$:



quantum jumps

In the following, we consider a simple concrete example. The atom has only **two** internal states, i.e., $\mathfrak{H}_A = \mathbb{C}^2$, and its Hamiltonian is given by

$$H_A := \hbar\omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \tag{17}$$

The time evolution of ensemble states

The states of the atom are described by 2×2 matrices of the form

$$\Omega \equiv \Omega(\vec{n}) := \frac{1}{2}\big[\mathbf{1}_2 + \vec{n} \cdot \vec{\sigma}\big], \quad \vec{n} \in \mathbb{R}^3, \text{ with } |\vec{n}| \leq 1, \tag{18}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) =$ vector of Pauli matrices. The state $\Omega(\vec{n})$ is **pure** iff $\vec{n}$ is a unit vector, i.e., $\vec{n}$ lies on the so-called *Bloch sphere;* and

$$\Omega(\vec{n}) + \Omega(-\vec{n}) = \mathbf{1}_2 .$$

We introduce lowering- and raising operators

$$\sigma_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ resp.}$$

In order to describe the *fluorescence* of such an atom, one sets

$$\mathfrak{L}_\alpha[\Omega] := -i\hbar^{-1}\big[H_A, \Omega\big] + \alpha\big[\sigma_- \Omega\, \sigma_+ - \frac{1}{2}\big\{\Omega, \sigma_+\sigma_-\big\}\big] . \tag{19}$$

The Lindblad equation can then be written as a **linear equation** for the vector $\vec{n}$ in the unit ball of $\mathbb{R}^3$:

Unraveling the Lindblad evolution

$$\dot{\vec{n}}(t) = \underbrace{\frac{\omega}{2}\,\vec{e}_3 \wedge \vec{n}(t)}_{\text{precession around } \vec{e}_3} - \underbrace{\frac{\alpha}{4}\Big[2\vec{e}_3 + \vec{n}(t) + n_3(t)\cdot\vec{e}_3\Big]}_{\text{dissipation}}. \qquad (20)$$

We observe that $\vec{n}(t) \to -\vec{e}_3$, as $t \to \infty$.
We now "unravel" Eq. (20) by specializing the procedure of Sect. 3 to the present model: Let $\vec{n}(t)$ be a unit vector, and let $\bar{\vec{n}}(t+dt)$ be given by

$$\bar{\vec{n}}(t+dt) = \vec{n}(t) + \Big\{\frac{\omega}{2}\,\vec{e}_3 \wedge \vec{n}(t) - \frac{\alpha}{4}\Big[2\vec{e}_3 + \vec{n}(t) + n_3(t)\cdot\vec{e}_3\Big]\Big\}dt\,.$$

The principles of the *ETH - Approach* then imply that $\bar{\vec{n}}(t+dt)$ must be replaced by a unit vector $\vec{n}(t+dt)$ whose Law is given by (see figure!)

$$\begin{aligned}
\vec{n}(t+dt) &= \frac{\bar{\vec{n}}(t+dt)}{\big|\bar{\vec{n}}(t+dt)\big|}, \qquad \text{with probability} \quad p_{nj}[t, t+dt], \\
\vec{n}(t+dt) &= -\frac{\bar{\vec{n}}(t+dt)}{\big|\bar{\vec{n}}(t+dt)\big|}, \qquad \text{with probability} \quad p_{flip}[t, t+dt]\,,
\end{aligned} \qquad (21)$$

The Poisson flip process on the Bloch sphere

where

$$p_{nj}[t, t+dt] = \frac{1 + \left|\vec{\overline{n}}(t+dt)\right|}{2} = 1 - \mathcal{O}(dt)$$
$$p_{flip}[t, t+dt] = \frac{1 - \left|\vec{\overline{n}}(t+dt)\right|}{2} = \mathcal{O}(dt)\,, \tag{22}$$

hence $p_{nj}[t, t+dt] + p_{flip}[t, t+dt] = 1$, as required.

Theorem: *Suppose there isn't any flip in the time interval* $[t_1, t_2)$*, with* $\vec{n}(t = t_1)$ *a given unit vector. Then, for* $t \in [t_1, t_2)$*,* $\vec{n}(t)$ *has the form*

$$\vec{n}(t) = \begin{pmatrix} \sqrt{1 - n_3(t)^2}\cos(\omega t + \gamma) \\ \sqrt{1 - n_3(t)^2}\sin(\omega t + \gamma) \\ n_3(t) \end{pmatrix}\,,$$

for some constant γ*, where* $n_3(t)$ *solves the* **non-linear** *equation*

$$\dot{n}_3(t) = -\frac{\alpha}{4}\Big(1 + n_3(t)\Big)\Big(1 - n_3(t)\Big)\Big(2 + n_3(t)\Big)\,. \tag{23}$$

Consequences of (23), and formulae for p_{nj}

- If $n_3(t_1) = -1$, i.e., atom in *ground-state,* then $n_3(t) \equiv -1$, and

$$p_{nj}[t_1, t] \equiv 1, \; \forall \; t > t_1 \, .$$

- If $n_3(t_1) = +1$, i.e., atom in *excited state,* then $n_3(t) = +1$, for $t \in [t_1, t_2)$, followed by transition to *ground-st.* at time t_2, with

$$p_{nj}[t_1, t_2) = \exp[-\alpha \, (t_2 - t_1)] \qquad \text{(exponential decay law)} \, .$$

- If $n_3(t_1) \in (-1, 1)$ then the state variable $\vec{n}(t)$ of the atom precesses around $\vec{e}_3$ with angular frequency ω, and

$$\frac{d\ell n\big(p_{nj}[t_1, t]\big)}{dt} = -\frac{\alpha}{4}\Big(1 + n_3(t)\Big)^2 ,$$

whose solution is given by

$$p_{nj}[t_1, t] = \exp[-\lambda(t)], \text{ where } \lambda(t) = \frac{1}{2}\int_{n_3(t)}^{n_3(t_1)} \frac{1+\tau}{(1-\tau)(2+\tau)} d\tau \, .$$

7. Conclusions

Using these (very explicit) results and (22), state trajectories with flips at times $t_n + dt_n, n = 1, 2, \ldots$, can be treated as in Sect. 3.

- For sufficiently large times (after finitely many flips), the atom is always found in its ground-state; i.e., $\vec{n}(t) \longrightarrow -\vec{e}_3$, as $t \longrightarrow \infty$.

Conclusions: In idealized models of systems of charged matter interacting with the quantized radiation field, described in the limit where the velocity of light $c \to \infty$, the principles of the *ETH*-**Approach** to (or the "*ETH*-**Completion**" of) Quantum Mechanics apparently yield **totally explicit predictions** concerning the **stochastic time evolution** of states of **individual** systems. This non-linear evolution is given by a novel type of stochastic process, called **quantum Poisson jump process**, which I have described in this talk; (see also L. Diósi, J. Phys. A **21**, 2885 (1988) for earlier related, but conceptually problematic results).

It would be interesting to test the fine-print of these predictions in experiments. The mathematics of **quantum Poisson jump processes** (& of generaralizations thereof) deserves to be worked out more fully.

A new (dynamical) approach to *"quantum chaos"* is emerging.

Relativistic quantum theory?

Remarks:

- **Physics**. As indicated in this lecture, fields describing massless modes, in particular the **EM field**, are responsible for the stochastic nature of the evolution of states of individual systems.
- The methods, described in this lecture on the example of the theory of fluorescence of atoms, can also be used to describe **measurement processes** by using nothing but the basic principles of the *ETH-* **Approach**. (An outline of this has been published in a paper with A. Pizzo.)
- A variant of the *ETH-***Approach** applicable to **relativistic quantum theory** has been developed, too. (It deserves to be worked out more fully.)

Thanks for listening!