An Introduction to Causal Fermion Systems and the Causal Action Principle

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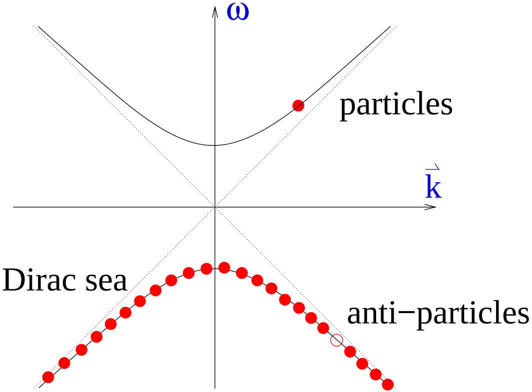
What is a causal fermion system?

- approach to fundamental physics
- novel mathematical model of spacetime
- physical equations are formulated in generalized spacetimes

starting point: Course on relativistic QM and QFT
(following Bjorken-Drell / Itzykson-Zuber)

- Dirac's hole theory (Dirac 1932)



How causal fermion systems developed ($\approx$ 1989-90)

- Problems of the naive Dirac sea picture:
  - infinite charge density
  - infinite negative energy density
- Therefore, we were told in lecture:
  - Dirac sea is not visible due to symmetries (homogeneous, isotropic)
  - Only “deviations” of the sea are observed as particles and anti-particles
  - Forget about the Dirac sea, no longer needed.
- This procedure is implemented in the formalism:
  - Reinterpretation of creation as annihilation operators
  - Wick ordering of field operators in Hamiltonian

I was not convinced by this procedure:

- The interacting Dirac sea should be visible,
  for example in presence of external potential

$$(i\partial\!\!\!/ + A\!\!\!/(x) - m)\psi = 0$$

- Pair creation seems an evidence that the Dirac sea is real.

How causal fermion systems developed ($\approx$ 1989-90)

What is the way out?

- Take all the sea states into account.
- In order to avoid divergences,
  formulate new type of equations,
  different structure of the physical equations

Goal in general terms:

Formulate a variational principle
directly for the ensemble of wave functions

- Intuitive picture: wave functions "organize themselves" in such a way that the Dirac sea configuration is a minimizer.
- In interacting situation the wave functions organize to solutions of the Dirac equation

$$(i\gamma^j \partial_j + e\gamma^j A(x) - m)\psi = 0$$

This should serve as the definition of A.

How causal fermion systems developed ($\approx$ 1990-91)

First attempts:
describe the "ensemble of all wave functions" by

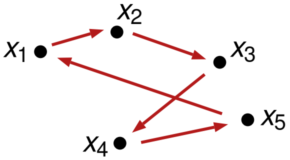
$$P(x,y) = -\sum_a \psi_a(x)\,\overline{\psi_a(y)} \quad \text{kernel of fermionic projector}$$

- gauge phases under local gauge transformations

$$A_j(x) \to A_j(x) + \partial_j \Lambda\,, \qquad \psi(x) \to e^{-i\Lambda(x)}\,\psi(x)$$
$$P(x,y) \to e^{-i\Lambda(x)}\,P(x,y)\,e^{i\Lambda(y)}$$

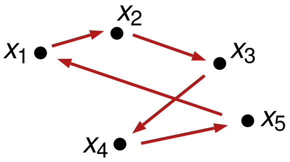
- gauge invariant: closed chain

$$A_{x_1\ldots x_N} := P(x_1,x_N)\cdots P(x_3,x_2)\,P(x_2,x_1) \;:\; S_{x_1} \to S_{x_1}$$



How causal fermion systems developed ($\approx$ 1990-91)

$$A_{x_1 \ldots x_N} := P(x_1, x_N) \cdots P(x_3, x_2)\, P(x_2, x_1) \;:\; S_{x_1} \to S_{x_1}$$



- Lagrangian $\mathcal{L}(A_{x_1 \ldots x_N})$ formed of eigenvalues of closed chain (e.g. polynomials of traces of powers)
- Form the action by integrating over spacetime points,

$$\mathcal{S} = \int_{\mathcal{M}} d^4x_1 \cdots \int_{\mathcal{M}} d^4x_N \, \mathcal{L}(A_{x_1 \ldots x_N})$$

- Seek for critical points (or minima) of the action.

How causal fermion systems developed ($\approx$ 1990-91)

Basic questions:

- Is this a sensible concept?
- How should the Lagrangian look like?

Guiding principles:

- vacuum Dirac sea configurations should be a critical point (or minimizer)
- the EL equations should
  reproduce the classical field equations
  (Maxwell+Einstein, later electroweak and strong)

A bit more technically:

$$P(x,y) = (i\gamma^j \partial_j + m)\, T_{m^2}(x,y)$$

with

$$\xi := y - x\,, \quad \xi^2 := \xi^i \xi_i$$

$$\begin{aligned} T_{m^2}(x,y) = &-\frac{1}{8\pi^3} \left(\frac{\mathrm{PP}}{\xi^2} + i\pi \delta\big(\xi^2\big)\, \epsilon\big(\xi^0\big) \right) \\ &+ \frac{m^2}{32\pi^3} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\, \frac{\big(m^2 \xi^2\big)^j}{4^j} \\ &\times \left(\log\left|\xi^2\right| + i\pi\, \Theta\big(\xi^2\big)\, \epsilon\big(\xi^0\big) + c_j \right) \end{aligned}$$

In the presence of an external potential (for example A EM),
light-cone expansion, Hadamard expansion in Minkowski space

$$
\begin{aligned}
\tilde{P}(x,y) = \frac{i}{2}\, \exp\Big(-i \int_0^1 A_j\big|_{\alpha y+(1-\alpha)x}\, \xi^j\, d\alpha \Big)\, P(x,y) \\
&- \frac{1}{2}\, \not{\xi}\, \xi_i \int_0^1 (\alpha-\alpha^2)\, j^i\big|_{\alpha y+(1-\alpha)x}\, d\alpha\;\; T^{(0)} \\
&+ \frac{1}{4}\, \not{\xi} \int_0^1 F^{ij}\big|_{\alpha y+(1-\alpha)x}\, \gamma_i\gamma_j\, d\alpha\;\; T^{(0)} \\
&- \xi_i \int_0^1 (1-\alpha)\, F^{ij}\big|_{\alpha y+(1-\alpha)x}\, \gamma_j\, d\alpha\;\; T^{(0)} \\
&- \xi_i \int_0^1 (1-\alpha)(\alpha-\alpha^2)\, \not{\partial} j^i\big|_{\alpha y+(1-\alpha)x}\, d\alpha\;\; T^{(1)} \\
&- \int_0^1 (1-\alpha)^2\, j^i\big|_{\alpha y+(1-\alpha)x}\, \gamma_i\, d\alpha\;\; T^{(1)} \\
&+ \not{\xi}\, (\deg < 1) + (\deg < 0) + O(A^2)
\end{aligned}
$$

How causal fermion systems developed ($\approx$ 1991-06)

By $\approx$ 1996 (two arXiv preprints)

- Only closed chain with two points make sense, i.e.

$$A_{xy} = P(x,y)\, P(y,x)$$

(otherwise electromagnetic flux comes into play)

- One needs to introduce an ultraviolet regularization,

$$A_{xy} = P^{\varepsilon}(x,y)\, P^{\varepsilon}(y,x)$$

Around 2001 concrete proposal for Lagrangian (see book from 2006):

$$\mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \Big(\big|\lambda_i^{xy}\big| - \big|\lambda_j^{xy}\big| \Big)^2$$

where λ_i^{xy} with $i = 1, \ldots, 4$ are eigenvalues of A_{xy}

Why do absolute values of the eigenvalues come up?

- phases of eigenvalues λ_i^{xy} drop out
  (crucial for getting chiral gauge field in the continuum limit)
- connection to causality: Consider Minkowski vacuum,
  $\xi := y - x$ not lightlike

$$P(x,y) = \alpha\, \xi_j \gamma^j + \beta\, \mathbb{1} \quad \text{(Lorentz symmetry)}$$
$$P(y,x) = \overline{\alpha}\, \xi_j \gamma^j + \overline{\beta}\, \mathbb{1}$$
$$A_{xy} = P(x,y)\, P(y,x) = a\, \xi_j \gamma^j + b\, \mathbb{1}$$
$$a = \alpha \overline{\beta} + \beta \overline{\alpha}\,, \qquad b = |\alpha|^2\, \xi^2 + |\beta|^2 \;\in\; \mathbb{R}$$

Applying the formula $(A_{xy} - b\mathbb{1})^2 = a^2\, \xi^2\, \mathbb{1}$, the eigenvalues of A_{xy} are computed by

$$\lambda_i^{xy} = b \pm |a| \sqrt{(y-x)_j (y-x)^j}$$

$$\begin{cases} \text{are real} & \text{if } y-x \text{ timelike} \\ \text{form complex conjugate pair} & \text{if } y-x \text{ spacelike} \end{cases}$$

How causal fermion systems developed ($\approx$ 2006)

Status of the theory:

- Usual setup: Minkowski space (causal and metric structures, Dirac matrices, Dirac wave functions, electromagnetic potential, ...)
- New action principle:

$$P(x,y) = -\sum_a \psi_a(x)\,\overline{\psi_a(y)}$$

$$\text{minimize} \quad \mathcal{S} = \int_{\mathscr{M}} d^4x \int_{\mathscr{M}} d^4y\, \mathcal{L}(x,y)$$

under variations of the wave functions ψ_a

This is conceptually unclear, too many structures, some of them appear twice.

How causal fermion systems developed ($\approx$ 2006-11)

Which structures are essential?

- Drop all structures which are not needed for the formulation of the causal action principle:
  - Minkowski metric and its causal structure, ...
  - Dirac matrices, Dirac equation, ...
  - gauge potentials, Maxwell equations, ...
- Keep:
  - $\mathscr{M}$ topological space with measure μ (spacetime volume)
  - $S_x\mathscr{M}$ spinor space, endowed with indefinite inner product $\prec .|. \succ_x$ (topological vector bundle with fiber metric)
  - wave functions $\psi(x) \in S_x\mathscr{M}$ (sections of vector bundle)
  - Hilbert space structure, needed because:

$$P(x, y) = -\sum_a |\psi_a(x) \succ \prec \psi_a(y)|$$

here the ψ_a should be orthonormal (more details later).

All the dropped structures should be emergent.

How causal fermion systems developed ($\approx$ 2011)

This was first considered in a discrete spacetime
(skip here; see book from 2006)

Two constructions turned out to be very helpful:

$$P(x,y) = -\sum_a |\psi_a(x) \succ \prec \psi_a(y)| \quad \text{kernel of fermionic projector}$$

$$A_{xy} = P(x,y)\, P(y,x) \quad \text{closed chain}$$

A_{xy} is isospectral to $F(y)\, F(x)$ with

$$F(x)^a_b = -\prec \psi_a(x) | \psi_b(x) \succ \quad \text{local correlation operator}$$

advantages:

- gauge phases drop out, gauge-invariant formulation
- bundle structure no longer needed

How causal fermion systems developed ($\approx$ 2011)

Why are A_{xy} and $F(x)\, F(y)$ isospectral?

$$P(x,y) = -\sum_a |\psi_a(x) \succ \prec \psi_a(y)|, \qquad F(x)^a_b = -\prec \psi_a(x) | \psi_b(x) \succ$$

Consider traces of powers:

$$\begin{aligned}
\mathrm{Tr}\left(A_{xy}^p\right) &= -\sum_a \mathrm{Tr}\Big(|\psi_a(x) \succ \prec \psi_a(y)|\; P(y,x)\; A_{xy}^{p-1} \Big) \\
&= -\sum_a \prec \psi_a(y)|\; P(y,x)\; A_{xy}^{p-1} |\psi_a(x) \succ \\
&= \sum_{a,b} \prec F(y)^a_b \prec \psi_b(x) \,|\, A_{xy}^{p-1} |\psi_a(x) \succ \\
&= \cdots = \mathrm{Tr}\Big(F(y)\, F(x) \cdots F(y)\, F(x) \Big)
\end{aligned}$$

How causal fermion systems developed ($\approx$ 2011)

Final simplification: Get rid of spacetime $\mathscr{M}$ (Minkowski space)

$$F(x)^a_b = -\prec\!\psi_a(x)|\psi_b(x)\!\succ \qquad \text{local correlation operator}$$

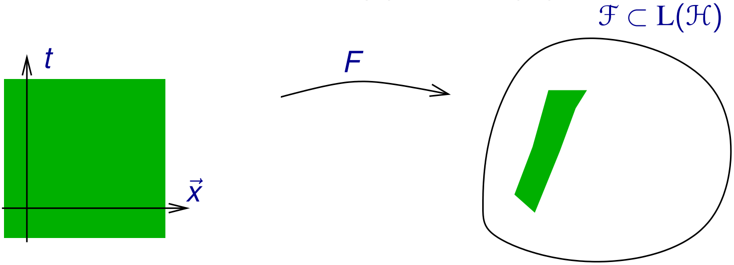
- F is symmetric and has rank ≤ 4
- F has at most 2 positive

  and at most 2 negative eigenvalues

denote such operators by $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$

How causal fermion systems developed ($\approx$ 2011)

We obtain mapping $\quad x \mapsto F(x) \in \mathcal{F} \subset \mathrm{L}(\mathcal{H})$



- push-forward measure $\rho := F_* \mu_{\mathscr{M}}$, is measure on $\mathcal{F}$,

$$\rho(\Omega) := \mu_{\mathscr{M}}\big(F^{-1}(\Omega)\big)\,, \qquad d\mu_{\mathscr{M}} = d^4x$$

- image of F recovered as the support of the measure,

$$M := \operatorname{supp} \rho = \Big\{ F \in \mathcal{F} \;|\; \rho(\Omega) \neq 0 \text{ for every open neighborhood } \Omega \text{ of } x \Big\}$$

Causal fermion systems (2011)

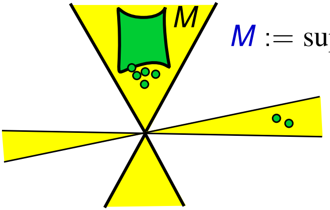
Definition (Causal fermion system)

Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space
Given parameter $n \in \mathbb{N}$ ("spin dimension")

$$\mathcal{F} := \Big\{ x \in \mathrm{L}(\mathcal{H}) \text{ with the properties:}$$

- x is self-adjoint and has finite rank
- x has at most n positive and at most n negative eigenvalues $\Big\}$

ρ a measure on $\mathcal{F}$.



$M := \operatorname{supp} \rho$ spacetime

Causal action principle

Let $x, y \in \mathcal{F}$. Then x and y are linear operators.

$x \cdot y \in \mathrm{L}(H)$:

- rank $\leq 2n$
- in general not self-adjoint: $(x \cdot y)^* = y \cdot x \neq x \cdot y$

thus non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Causal action principle

Nontrivial eigenvalues of xy: $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$

Lagrangian $$\mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \Big(|\lambda_i^{xy}| - |\lambda_j^{xy}| \Big)^2 \;\geq\; 0$$

action $$\mathcal{S} = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, d\rho(x)\, d\rho(y) \;\in\; [0,\infty]$$

Minimize $\mathcal{S}$ under variations of ρ, with constraints

volume constraint: $$\rho(\mathcal{F}) = \text{const}$$

trace constraint: $$\int_{\mathcal{F}} \text{tr}(x)\, d\rho(x) = \text{const}$$

boundedness constraint: $$\iint_{\mathcal{F}\times\mathcal{F}} \sum_{i=1}^{2n} |\lambda_i^{xy}|^2\, d\rho(x)\, d\rho(y) \leq C$$

- F.F., "Causal variational principles on measure spaces," *J. Reine Angew. Math.* **646** (2010) 141–194

A few remarks

One basic object: measure ρ on set $\mathcal{F}$ of linear operators on $\mathcal{H}$, describes spacetime as well as all objects therein

- Underlying structure: ensemble of fermionic wave functions (more details later)
- Geometric structures encoded in these wave functions

  Matter encodes geometry
  Quantum spacetime

- Causal action principle describes spacetime as a whole (similar to Einstein-Hilbert action in GR)
- Causal action principle is a nonlinear variational principle (similar to Einstein-Hilbert action or classical field theory)
- Linear dynamics of Quantum theory recovered in limiting case (more details later, in connection of collapse)

Underlying physical principles

- local gauge principle:
  freedom to perform local unitary transformations of the spinors
- Pauli exclusion principle:
  Choose orthonormal basis $\psi_1, \ldots, \psi_f$ of $\mathcal{H}$. Set

  $$\Psi = \psi_1 \wedge \cdots \wedge \psi_f \,,$$

  gives equivalent description by Hartree-Fock state.
- the “equivalence principle”:
  symmetry under “diffeomorphisms” of M
  (note: M merely is a topological measure space)

Spacetime and causal structure are emergent

Interpretation in terms of spacetime events

- operators in $\mathcal{F}$ can be interpreted as "possible local correlation operators"
  or simply as possible events
- operators in M are the events realized in spacetime
- spacetime is made up of all the realized events
- the physical equations relate the events to each other

For details on this connection:

- F.F, J. Fröhlich, C. Paganini, C. and M. Oppio, "Causal fermion systems and the ETH approach to quantum theory," arXiv:2004.11785 [math-ph] (2020) www.causal-fermion-system.com

Thank you for your attention!

The Continuum Limit of Causal Fermion Systems and Quantum States

Inherent structures of a causal fermion system

Let $(\rho, \mathcal{F}, \mathcal{H})$ be a causal fermion of spin dimension n, spacetime $M := \text{supp}\,\rho$.

spacetime points are linear operators on $\mathcal{H}$

- For $x \in M$, consider eigenspaces of x.
- For $x, y \in M$,
  - consider operator products xy
  - project eigenspaces of x to eigenspaces of y

Gives rise to:

- quantum objects (spinors, wave functions)
- geometric structures (connection, curvature)
- causal structure, analytic structures

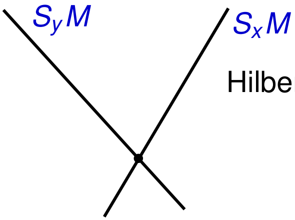
Inherent structures of a causal fermion system

- Spinors

$$S_xM := x(\mathcal{H}) \subset \mathcal{H} \qquad \text{“spin space”}, \ \dim S_xM \leq 2n$$

$$\prec u|v\succ_x := -\langle u \,|\, x\, v\rangle_{\mathcal{H}} \qquad \text{“spin scalar product”},$$

inner product of signature $(\leq n, \leq n)$



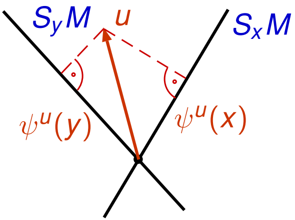
Hilbert space $\mathcal{H}$

Inherent structures in spacetime

- Physical wave functions

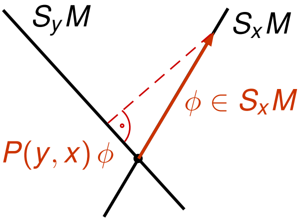
$$\psi^u(x) = \pi_x \, u \;\text{ with } u \in \mathcal{H} \qquad \text{physical wave function}$$

$$\pi_x : \mathcal{H} \to \mathcal{H} \qquad \text{orthogonal projection on } x(\mathcal{H})$$



- The kernel of the fermionic projector:

$$P(y,x) = \pi_y \, x|_{S_x M} \;:\; S_x M \to S_y M$$



$$P(y,x) = -\sum_{i=1}^{f} |\psi^{e_i}(y) \succ \prec \psi^{e_i}(x)| \qquad \text{where } (e_i) \text{ ONB of } \mathcal{H}$$

Inherent structures in spacetime

- **Geometric structures**
  - $P(x,y) \;:\; S_yM \rightarrow S_xM$ yields relations between spin spaces.
    Using a polar decomposition $(\ldots, \ldots)$ one gets:

    $$D_{x,y} \;:\; S_yM \rightarrow S_xM \text{ unitary} \qquad \text{“spin connection”}$$

  - tangent space T_x, carries Lorentzian metric,

    $$\nabla_{x,y} \;:\; T_y \rightarrow T_x \qquad \text{corresponding “metric connection”}$$

  - holonomy of connection gives curvature

    $$R(x,y,z) = \nabla_{x,y}\,\nabla_{y,z}\,\nabla_{z,x} \;:\; T_x \rightarrow T_x$$

Causal structure

Let $x, y \in M$. Then
$x \cdot y \in \mathrm{L}(H)$ has non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Definition (causal structure)

The points $x, y \in \mathcal{F}$ are called

$$\begin{cases} \text{spacelike separated} & \text{if } |\lambda_j^{xy}| = |\lambda_k^{xy}| \text{ for all } j, k = 1, \ldots, 2n \\ \text{timelike separated} & \text{if } \lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \text{ are all real} \\ & \text{and } |\lambda_j^{xy}| \neq |\lambda_k^{xy}| \text{ for some } j, k \\ \text{lightlike separated} & \text{otherwise} \end{cases}$$

- Lagrangian is compatible with causal structure:

$$\text{Lagrangian} \quad \mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \geq 0$$

thus x, y spacelike separated $\quad \Rightarrow \quad \mathcal{L}(x,y) = 0$

"points with spacelike separation do not interact"

A distinguished time direction

$$x(\mathcal{H}) \subset \mathcal{H} \qquad \text{subspace of dimension} \leq 2n$$

Introduce the functional

$$\mathcal{C} \,:\, M \times M \to \mathbb{R}\,, \qquad \mathcal{C}(x,y) := i \,\mathrm{tr}\big(y\, x\, \pi_y\, \pi_x - x\, y\, \pi_x\, \pi_y\big)$$

For timelike separated points $x, y \in M$,

$$\begin{cases} y \text{ likes in the future of } x & \text{if } \mathcal{C}(x,y) > 0 \\ y \text{ likes in the past of } x & \text{if } \mathcal{C}(x,y) < 0 \end{cases}$$

- The resulting relation “lies in the future of” is not necessarily transitive.

The continuum limit

Causal fermion system

- abstract mathematical framework
- quantum geometry, causal action

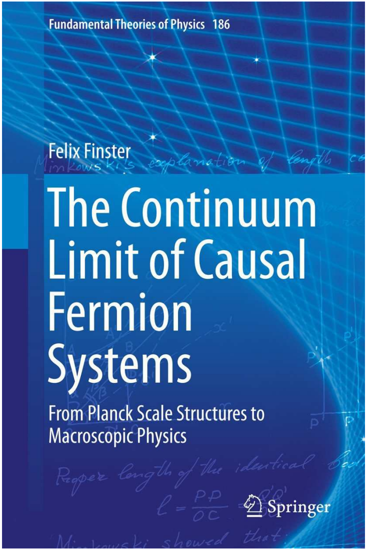
$\Downarrow$ continuum limit

description in the continuum limit

- Dirac fields
- strong and electroweak gauge fields
- gravitational field

- fermion field: second-quantized
- bosonic field: classical

The continuum limit



Fundamental Theories of Physics **186**

Springer, 2016

548+xi pages

arXiv:1605.04742 [math-ph]

The causal action principle in the continuum limit

- specify vacuum as sum of Dirac seas,

$$P(x,y) = \sum_{\beta=1}^{g} P^{\text{sea}}_{m_\beta}(x,y)$$

$$P^{\text{sea}}_{m_\beta}(x,y) = \int \frac{d^4k}{(2\pi)^4}\, (\slashed{k} + m_\beta)\, \delta(k^2 - m_\beta^2)\, \Theta(-k^0)\, e^{-ik(x-y)}$$

β labels "generations" of elementary particles

$\Longrightarrow$ Dynamical equations only if three generations ($g = 3$)

The causal action principle in the continuum limit

- Model involving neutrinos and quarks:

$$P(x,y) = \sum_{\beta=1}^{3} \underbrace{P^{\text{sea}}_{m_\beta}(x,y) \oplus \cdots \oplus P^{\text{sea}}_{m_\beta}(x,y)}_{\text{7 identical direct summands}} \oplus P^{\text{sea}}_{\tilde{m}_\beta}(x,y)$$

  - again three generations
  - $4 \times 8 = 32$-component wave functions
  - spin dimension 16
- Regularize on the scale ε (Planck scale), regularization of neutrinos breaks chiral symmetry

The causal action principle in the continuum limit

Remarks on methods for analyzing the continuum limit:

- Consider the Dirac equation in an external potential

$$(i\partial\!\!\!/ + \mathcal{B} - mY)\psi = 0 \,.$$

- Question: Are the EL equations of causal action principle satisfied in the limit $\varepsilon \searrow 0$?
- Answer: Yes if and only if $\mathcal{B}$ has a certain structure and satisfies the classical field equations.

Going beyond the continuum limit

Basic question: What about objects in space (densities, probabilities, etc.)

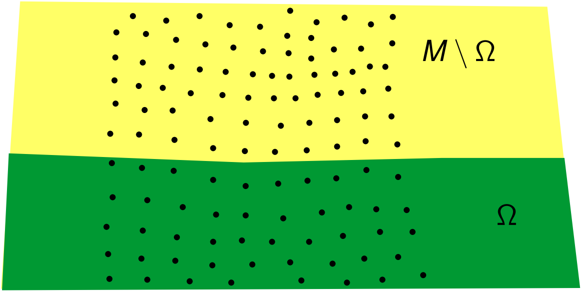
- The scalar product $\langle .|. \rangle_{\mathcal{H}}$ is given abstractly
- Missing: Representation as spatial integral.
  Is there an analog of the relation

$$\langle \psi | \phi \rangle_{\mathcal{H}} = \int_{\mathcal{N}} \prec \psi | \phi \succ_x \, d\mu_{\mathcal{N}}(x)$$

  for Dirac spinors?
- Related question: What about current conservation? Are there conservation laws?
- Ultimately: What is the quantum state?
  What are quantum probabilities?

Surface Layer Integrals

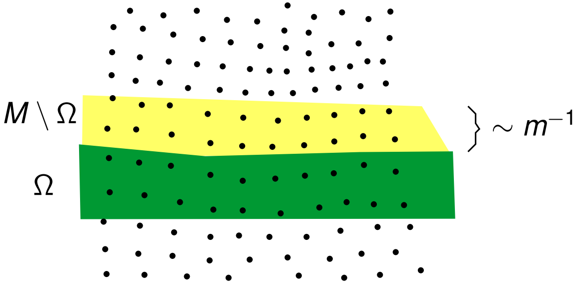
General structure of a surface layer integral:



$$\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, (\cdots)\mathcal{L}(x,y)$$

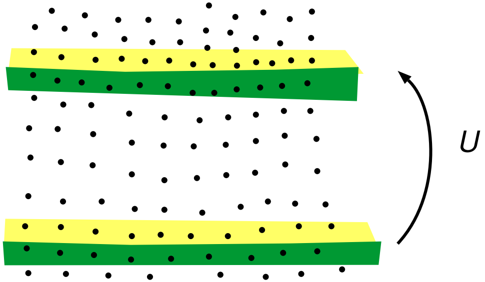
Surface Layer Integrals

Typically: $\mathcal{L}(x,y)$ very small if x and y far apart
(Compton scale)



- F.F., J. Kleiner, “Noether-like theorems for causal variational principles,” arXiv:1506.09076 [math-ph], *Calc. Var. Partial Differential Equations* **55:35** (2016)

Conservation laws for surface layer integrals



- F.F., J. Kleiner, "A class of conserved surface layer integrals for causal variational principles," arXiv:1801.08715 [math-ph], *Calc. Var. Partial Differential Equations* **58:38** (2019)
- F.F., N. Kamran, "Complex structures on jet spaces and bosonic Fock space dynamics for causal variational principles," arXiv:1808.03177 [math-ph], *Pure Appl. Math. Q.* **17** (2021) 55–140

The Euler-Lagrange equations

For clarity of presentation: leave out trace and boundedness constraints

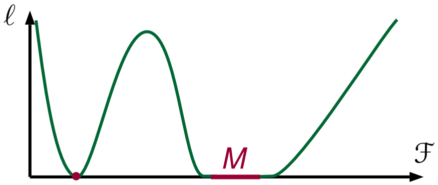
$$\ell(x) := \int_{\mathcal{F}} \mathcal{L}(x,y)\, d\rho(y) - \mathfrak{s}$$

($\mathfrak{s} > 0$ Lagrange multiplier for volume constraint)

Lemma

Let ρ be a minimizer of the causal action. Then

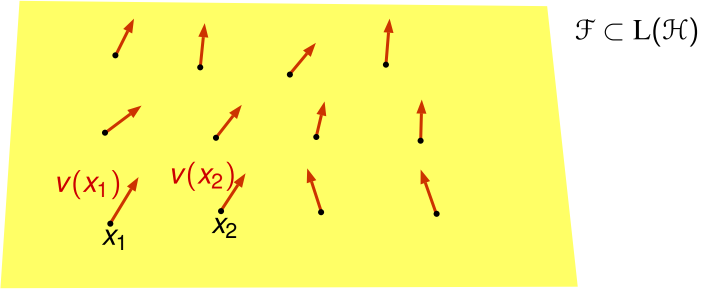
$$\ell|_M \equiv \inf_{\mathcal{F}} \ell = 0$$



Linear perturbations

To simplify presentation assume that:
ρ discrete minimizing measure describing the vacuum.

- What are linear perturbations of the measure?



Also a scalar weight function $b(x)$ comes into play

- jet $\mathfrak{v} := (b, v) \in \mathfrak{J}$

Jet dynamics

The jet $\mathfrak{v} = (b, v)$ satisfies the linearized field equations

$$\begin{aligned} 0 &= \langle \mathfrak{u}, \Delta \mathfrak{v} \rangle(x) \\ &:= \nabla_{\mathfrak{u}} \bigg(\int_M \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}} \big) \mathcal{L}(x,y)\, d\rho(y) - \nabla_{\mathfrak{v}}\, \mathfrak{s} \bigg) \end{aligned}$$

for all test jets $\mathfrak{u}$, where

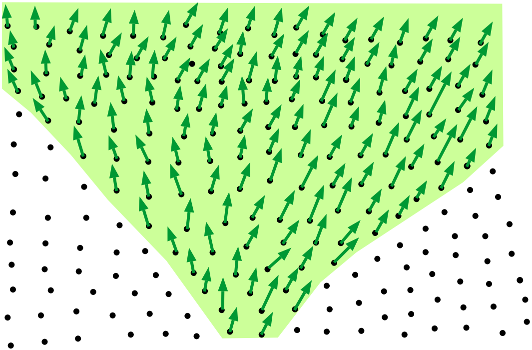
$$\nabla_{\mathfrak{v}} g(x) := a(x)\, g(x) + \big(D_v g \big)(x)$$

There are also corresponding nonlinear field equations.

- F.F., J. Kleiner, "A Hamiltonian formulation of causal variational principles," arXiv:1612.07192 [math-ph], *Calc. Var. Partial Differential Equations* **56:73** (2017)
- F.F., "Perturbation theory for critical points of causal variational principles," arXiv:1703.05059 [math-ph] (2017), *Adv. Theor. Math. Phys.* **24** (2020) 563–619

Existence, Uniqueness, Finite Propagation Speed

for linearized fields



This holds "on the macroscopic scale"

- C. Dappiaggi, F.F., "The Cauchy problem and the causal structure of linearized fields for causal variational principles," arXiv:1811.10587 [math-ph], *Methods Appl. Anal.* **27** (2020) 1–56

based on energy estimates

Surface layer integrals for linearized fields

- conserved surface layer integrals:

$$\gamma^\Omega_\rho \;:\; \mathfrak{J} \to \mathbb{R} \qquad \text{(conserved one-form)}$$

$$\gamma^\Omega_\rho(\mathfrak{u}) = \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, \big(\nabla_{1,\mathfrak{u}} - \nabla_{2,\mathfrak{u}}\big)\mathcal{L}(x,y)$$

$$\sigma^\Omega_\rho \;:\; \mathfrak{J}\times\mathfrak{J} \to \mathbb{R} \qquad \text{(symplectic form)}$$

$$\sigma^\Omega_\rho(\mathfrak{u},\mathfrak{v}) = \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, \big(\nabla_{1,\mathfrak{u}}\nabla_{2,\mathfrak{v}} - \nabla_{2,\mathfrak{u}}\nabla_{1,\mathfrak{v}}\big)\mathcal{L}(x,y)$$

- other useful surface layer integral
(conserved in non-interacting case)

$$(.,.)^\Omega_\rho \;:\; \mathfrak{J}\times\mathfrak{J} \to \mathbb{R} \qquad \text{(surface layer inner product)}$$

$$(\mathfrak{u},\mathfrak{v})^\Omega_\rho = \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, \big(\nabla_{1,\mathfrak{u}}\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{u}}\nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y)$$

give rise to complex structure

Surface layer integral for wave functions

Unitary invariance of causal action principle,

$$\mathcal{U}_\tau := \exp(i\tau\mathcal{A}) \qquad \text{with} \quad \mathcal{A}\psi := \langle u|\psi\rangle_{\mathcal{H}}\, u$$

described infinitesimally by commutator jets

$$\mathfrak{C} := (0, \mathcal{C}) \qquad \text{with} \qquad \mathcal{C}(x) := i\big[\mathcal{A}, x\big]$$

$$\gamma_\rho^\Omega(\mathfrak{C}) = \langle u|u\rangle_\rho^\Omega \quad \text{extended to}$$

$$\langle . | . \rangle_\rho^\Omega \;:\; \mathcal{W}_\rho^\Omega \times \mathcal{W}_\rho^\Omega \to \mathbb{C} \qquad \text{(commutator inner product)}$$

$$\langle \psi|\phi\rangle_\rho^\Omega = -2i \bigg(\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) - \int_{M\setminus\Omega} d\rho(x) \int_\Omega d\rho(y) \bigg) \times \prec\!\psi(x) \,|\, Q^{\mathrm{dyn}}(x,y)\, \psi(y)\!\succ_x \,.$$

- conserved if ψ, ϕ satisfy dynamical wave equation
- F.F., N. Kamran, M. Oppio, "The linear dynamics of wave functions in causal fermion systems," arXiv:2101.08673 [math-ph], *J. Differential Equations* **293** (2021) 115–187

General setting:

- Two minimizing causal fermion systems
  - $(\mathcal{H}, \mathcal{F}, \rho)$ describing vacuum
  - $(\tilde{\mathcal{H}}, \tilde{\mathcal{F}}, \tilde{\rho})$ describing the interacting spacetime
  - corresponding spacetimes:

$$M := \text{supp}\, \rho\,, \quad \tilde{M} := \text{supp}\, \tilde{\rho}$$

- Goal: Compare $\tilde{\rho}$ and ρ at time t.
- Basic object: Nonlinear surface layer integral
  - identify Hilbert spaces by choosing $V : \mathcal{H} \to \tilde{\mathcal{H}}$ unitary

$$\begin{aligned}\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \rho) := &\int_{\tilde{\Omega}} d\tilde{\rho}(x) \int_{M \setminus \Omega} d\rho(y)\, \mathcal{L}(x,y) \\ &- \int_{\tilde{M} \setminus \tilde{\Omega}} d\tilde{\rho}(x) \int_{\Omega} d\rho(y)\, \mathcal{L}(x,y)\end{aligned}$$

Freedom in identifying the Hilbert spaces

- identification of Hilbert spaces:
  - Choose $V : \mathcal{H} \to \tilde{\mathcal{H}}$ unitary
  - Work exclusively in $\mathcal{H}$
  - But: identification is not canonical, gives freedom

$$\rho \to \mathcal{U}\rho\,, \quad (\mathcal{U}\rho)(\Omega) := \rho(\mathcal{U}^{-1}\Omega\mathcal{U})$$

- This freedom is treated by integrating over $\mathcal{U}$
  - Let $\mathcal{G} \subset \mathrm{U}(\mathcal{H})$ be compact subgroup
  - $\mu_{\mathcal{G}}$ normalized Haar measure on $\mathcal{G}$

- symmetrized nonlinear surface layer integral

$$\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\mathcal{U}\rho) = \int_{\tilde{\Omega}} d\tilde{\rho}(x) \int_{M\setminus\Omega} d\rho(y)\, \mathcal{L}\big(x, \mathcal{U}y\mathcal{U}^{-1}\big) - \int_{\Omega} d\rho(x) \int_{\tilde{M}\setminus\tilde{\Omega}} d\tilde{\rho}(y)\, \mathcal{L}\big(x, \mathcal{U}y\mathcal{U}^{-1}\big)$$

$$\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\rho) = \fint_{\mathcal{G}} \gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\mathcal{U}\rho)\, d\mu_{\mathcal{G}}(\mathcal{U})$$

  can be arranged to vanish for all t (Greene-Shiohama)
- *partition function*

$$Z(\beta,\tilde{\rho}) = \fint_{\mathcal{G}} \exp\Big(\beta\, \gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\mathcal{U}\rho)\Big)\, d\mu_{\mathcal{G}}(\mathcal{U})$$

  where β free parameter (maybe discuss at the end)

How to “test” the interacting spacetime?

- Interacting spacetime can be arbitrarily complicated (interacting quantum fields, entanglement, collapse)
- describe by objects in the vacuum spacetime: free fields, wave functions, ...
- use insertions:

$$\frac{1}{Z^t} \int_{\mathcal{G}} (\cdots) \exp\Big(\beta\,\gamma^t(\tilde{\rho}, \mathcal{U}\rho)\Big)\, d\mu_{\mathcal{G}}(\mathcal{U})$$

  - formal analogy to path integral formalism

Field Operators in the Vacuum

- Canonical commutation/anti-commutation relations for $z, z' \in \mathfrak{h}$ and $\psi, \psi' \in \mathcal{H}^{\mathrm{f}}_\rho \subset \mathcal{H}_\rho$

$$\begin{aligned} \big[a(\overline{z}), a^\dagger(z')\big] &= (z|z')^\Omega_\rho \\ \big[a(\overline{z}), a(\overline{z'})\big] &= 0 = \big[a^\dagger(z), a^\dagger(z')\big] \\ \big\{\Psi(\overline{\phi}), \Psi^\dagger(\phi')\big\} &= \langle \phi|\phi'\rangle^\Omega_\rho \\ \big\{\Psi(\overline{\phi}), \Psi(\overline{\phi'})\big\} &= 0 = \big\{\Psi^\dagger(\phi), \Psi^\dagger(\phi')\big\} \end{aligned}$$

  - independent of time
  - generate unital $*$-algebra $\mathscr{A}$

Construction of the Quantum State

- Quantum state ω^t at time t:

$$\omega^t : \mathscr{A} \to \mathbb{C} \qquad \text{linear and positive, i.e.}$$
$$\omega^t(A^* A) \geq 0 \quad \text{for all } A \in \mathscr{A}$$

- More concretely, represented on Fock space:
  - With a density operator:

$$\omega^t(A) = \mathrm{tr}_{\mathcal{F}}\big(\sigma^t A\big)$$

  - As an expectation value (pure state):

$$\omega^t(A) = \langle \Psi | A | \Psi \rangle_{\mathcal{F}}$$

- General structure:

$$\omega^t(\cdots) := \frac{1}{Z} \int_{\mathcal{G}} (\cdots)\, e^{\beta\, \gamma^{\tilde{\Omega},\Omega}\left(\tilde{\rho},\mathcal{U}\rho\right)}\, d\mu_{\mathcal{G}}(\mathcal{U})$$

How do the insertions look like?

Definition of the state

DEFINITION

The state ω^t at time t is defined by

$$\omega^t\Big(a^\dagger(z'_1)\cdots a^\dagger(z'_p)\,\Psi^\dagger(\phi'_1)\cdots\Psi^\dagger(\phi'_{r'})$$

$$\times\; a(\overline{z_1})\cdots a(\overline{z_q})\,\Psi(\overline{\phi_1})\cdots\Psi(\overline{\phi_r})\Big)$$

$$:= \frac{1}{Z(\beta,\tilde{\rho})}\,\delta_{r'r}\,\frac{1}{p!}\sum_{\sigma,\sigma'\in S_r}(-1)^{\text{sign}(\sigma)+\text{sign}(\sigma')}$$

$$\times\int_{\mathcal{G}}\langle\tilde{\phi}_{\sigma(1)}\,|\,\pi^t_{\mathcal{U}}\,\tilde{\phi}'_{\sigma'(1)}\rangle^\Omega_\rho\cdots\langle\tilde{\phi}_{\sigma(r)}\,|\,\pi^t_{\mathcal{U}}\,\tilde{\phi}'_{\sigma'(r)}\rangle^\Omega_\rho$$

$$\times\; D_{\tilde{z}'_1}\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\mathcal{U}\rho)\cdots D_{\tilde{z}'_p}\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\mathcal{U}\rho)$$

$$\times\; D_{\overline{\tilde{z}}_1}\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\mathcal{U}\rho)\cdots D_{\overline{\tilde{z}}_q}\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\mathcal{U}\rho)\;e^{\beta\,\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\mathcal{U}\rho)}\;d\mu_{\mathcal{G}}(\mathcal{U})$$

Positivity of the Quantum State

THEOREM

The state ω^t is positive, i.e.

$$\omega^t\big(A^*A\big) \geq 0 \qquad \textit{for all } t \in \mathbb{R} \textit{ and } A \in \mathscr{A}$$

The proof makes use of

- Canonical commutation/anti-commutation relations
- Positivity of $(.|.)^\Omega_\rho$ and $\langle .|.\rangle^\Omega_\rho$
- Positivity of insertions:

$$D_{\tilde{z}}\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\mathfrak{U}\rho) \cdot D_{\overline{\tilde{z}}}\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\mathfrak{U}\rho) = \big|D_{\tilde{z}}\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho},\mathfrak{U}\rho)\big|^2 \geq 0$$

$$\langle \psi \,|\, \pi^t_{\mathfrak{U}}\, \psi\rangle^\Omega_\rho \geq 0 \quad \text{and} \quad \langle \psi \,|\, (\mathbb{1} - \pi^t_{\mathfrak{U}})\, \psi\rangle^\Omega_\rho \geq 0$$

- F.F. and Kamran, N., “Fermionic Fock spaces and quantum states for causal fermion systems,” arXiv:2101.10793 [math-ph], to appear in *Ann. Henri Poincaré* (2022)

Representations of the Quantum State

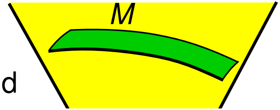
- GNS representation
  - Introduce scalar product on $\mathscr{A}$ by
    $$\langle A | A' \rangle := \omega^t(A^* A') \;:\; \mathscr{A} \times \mathscr{A} \to \mathbb{C}$$
    Forming the completion gives a Hilbert space.
  - $\mathscr{A}$ has a natural representation on this Hilbert space.
  - Setting $\Phi = \mathbb{1}$,
    $$\langle \Phi \,|\, A\, \Phi \rangle = \omega^t(\mathbb{1}^*\, A\, \mathbb{1}) = \omega^t(A)$$
  - always exists, but in general not a Fock representation
- Representation on the Fock space of vacuum
  - choose $\mathcal{F}$ as the Fock space generated by acting with $\mathscr{A}$ on vacuum state (Dirac sea vacuum)
  - construct density operator σ^t on $\mathcal{F}$ with
    $$\omega^t(A) = \mathrm{tr}_{\mathcal{F}}\big(\sigma^t\, A\big)$$
  - inductive construction for states involving *finite number of particles and anti-particles*
  - in general diverges (inequivalent Fock vacua, . . .)
  - makes connection to perturbative description

Outlook: Dynamics of the quantum state

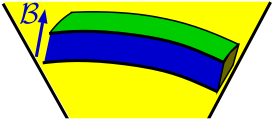
- Construction so far gives ω^t for all t
- Next steps:
  - Construct time evolution for the density operator
$$\mathfrak{L}^t_{t_0} \;:\; \sigma^{t_0} \to \sigma^t$$
  - Is there a unitary time evolution on the Fock space?
$$\omega^t = U^t_{t_0}\, \omega^{t_0}\, \big(U^t_{t_0}\big)^{-1}$$
- Is ongoing work with N. Kamran and M. Reintjes

Outlook: a quantum spacetime

a classical space-time:
M diffeomorphic to manifold



a quantum space-time:
$M \simeq \mathscr{M} \times \mathcal{B}$



- microscopic mixing, holographic mixing
- integrating over additional "degrees of freedom" $\mathcal{B}$ resembles path integral

  … …

- F.F., "Perturbative Quantum Field Theory in the Framework of the Fermionic Projector"
  arXiv:1310.4121 [math-ph], J. Math. Phys. **55** (2014) 042301

Outlook: Connection to collapse models

General structure:

- Nonlinear dynamics of $\tilde{\rho}$ (from causal action principle)
- Conservation laws hold
  (current conservation, conserved symplectic form, ...)
- Causality holds in the sense
  "pairs of points with spacelike separation do not interact"
  in particular: no superluminal signalling
- In approximation ("approximation of inhomogeneous fluctuating fields") one gets linear and unitary time evolution

$$U_{t_0}^t \;:\; \mathcal{F} \to \mathcal{F}$$

Outlook: Connection to collapse models

As observed by J. Kleiner, this seems to indicate that causal fermion systems are an effective collapse theory.

A. Bassi, D. Dürr, G. Hinrichs, *"Uniqueness of the equation for quantum state vector collapse,"* Phys. Rev. Lett. **111**, 210401 (2013)

- No faster-than-light signalling
- Time evolution Markovian and homogeneous in time

$\Longrightarrow$ collapse theory

Can this be adapted to causal fermion systems?

www.causal-fermion-system.com

Thank you for your attention!